BOREL STRUCTURE IN GROUPS AND THEIR DUALS[1]

BY
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Introduction. In the past decade or so much work has been done toward extending the classical theory of finite dimensional representations of compact groups to a theory of (not necessarily finite dimensional) unitary representations of locally compact groups. Among the obstacles interfering with various aspects of this program is the lack of a suitable natural topology in the "dual object"; that is in the set of equivalence classes of irreducible representations. One can introduce natural topologies but none of them seem to have reasonable properties except in extremely special cases. When the group is abelian for example the dual object itself is a locally compact abelian group. This paper is based on the observation that for certain purposes one can dispense with a topology in the dual object in favor of a "weaker structure" and that there is a wide class of groups for which this weaker structure has very regular properties.

If S is any topological space one defines a Borel (or Baire) subset of S to be a member of the smallest family of subsets of S which includes the open sets and is closed with respect to the formation of complements and countable unions. The structure defined in S by its Borel sets we may call the Borel structure of S. It is weaker than the topological structure in the sense that any one-to-one transformation of S onto S which preserves the topological structure also preserves the Borel structure whereas the converse is usually false. Of course a Borel structure may be defined without any reference to a topology by simply singling out an arbitrary family of sets closed with respect to the formation of complements and countable unions. Giving a set of mathematical objects a topology amounts to giving it a sufficiently space-like structure so that one can speak of certain of the objects being "near to" or "far away from" certain others. Giving it a Borel structure amounts to distinguishing a family of "well behaved" or "definable" subsets. In using the terms

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(1) In various stages of development and in varying degrees of detail the material in this paper has been included as part of each of the following: (a) Two lectures given in Paris in October 1954 under the auspices of the "Colloque Henri Poincaré." (b) A series of ten lectures on group representations given under the auspices of the Princeton University physics department in May 1955 and supported by the Eugene Higgins fund. (c) A course in group representations given during the 1955 summer quarter at the University of Chicago. (d) A paper presented by title at the 1955 summer meeting of the American Mathematical Society (Abstract 61-6-726 *t*). Mimeographed lecture notes of the University of Chicago course have been issued by the University of Chicago mathematics department and it is possible that the Centre Nationale des Recherches Scientifiques will publish a volume containing the texts of the lectures presented at the "Colloque Henri Poincaré."

"well behaved" and "definable" we have reference to the fact that work of the past few decades in the foundations of mathematics makes it clear that the concept of an arbitrary subset of a noncountable set is much less clear and straightforward than it at first seems. The subsets which can be explicitly described in some more or less concrete way form a very small subfamily of the whole. Moreover unless one restricts attention to such "accessible" sets or to sets differing from them in some negligible manner one has such results as the Banach Tarski paradox.

This paper originated in a specific problem in the theory of group representations; that of studying the relationship between the irreducible representations of a group and those of a normal subgroup. When the normal subgroup K is abelian (cf. [9]) the solution of the problem leans heavily on the fact that a very complete decomposition theory is available for the unitary representations of locally compact abelian groups. The recently developed direct integral decomposition theory for Hilbert spaces and representations furnishes a partial substitute but does not go quite far enough. It needs to be augmented by certain considerations which require introduction and use of a "natural" Borel structure in the dual object of K. Coincidentally certain other difficulties arising in the solution of the same problem lead also to a consideration of Borel structures and their properties. For example a certain auxiliary group which is known a priori only to have a certain kind of Borel structure has to be given a locally compact topology.

While the primary purpose of this paper is to supply the theorems about Borel structures needed for our paper on the group representation problem it has been written with a view to other applications as well. We feel that Borel structure has been too much neglected in favor of topological structure and that it can be studied to advantage in a number of contexts. In particular we shall publish elsewhere [13] an application to the cohomology theory of group extensions.

To a considerable extent the general theorems about Borel structures which we need are already to be found in the literature in only mildly disguised form. Many of them indeed are reformulations of theorems about Borel subsets of metric spaces to be found in Kuratowski's *Topologie* I. The first few sections of the present article are devoted to laying down our basic definitions and reformulating known theorems in terms of them. In addition we deduce some rather surprising corollaries about the existence and uniqueness of "natural" Borel structures. Then we turn to the particular case of groups and their duals. After refining Weil's converse of Haar's theorem on the existence of invariant measures and giving a measure theoretic characterization of closed subgroups we pass to the decomposition theory for representations of groups (and algebras). Treating only the separable case we show that the dual object always has a natural Borel structure which by virtue of being a quotient structure of a "good" structure is itself either a "good" structure or quite "bad." Having a "good" Borel structure seems to be very closely related to having no type II or type III representations in the sense of Murray and von Neumann. Perhaps the two are equivalent. At any rate assuming *both* a "good" Borel structure and only type I representations we obtain a decomposition theory fully as complete as that in the abelian case.

1. **Preliminary definitions.** Let S be a set. By a *Borel structure* in S we shall mean a family $\mathcal{B}$ of subsets of S such that S and the empty set are both in $\mathcal{B}$ and such that $\bigcup_{j=1}^{\infty} E_j$, $\bigcap_{j=1}^{\infty} E_j$ and $S-E_j$ are in $\mathcal{B}$ whenever $E_1, E_2, \cdots$ are in $\mathcal{B}$; in other words a σ-field of subsets of S. We prefer the term "Borel structure" to the more familiar "σ-field" because the former term does more to suggest our underlying viewpoint. A set S together with a Borel structure $\mathcal{B}$ in (or for) S we shall call a *Borel space*. We shall often suppress the $\mathcal{B}$ and speak simply of the Borel space S. We shall call the members of $\mathcal{B}$ the *Borel subsets* of S. Clearly for each family $\mathfrak{F}$ of subsets of a set S there is a unique smallest Borel structure for S which contains $\mathfrak{F}$. We shall call it the *Borel structure generated by* $\mathfrak{F}$ and shall call $\mathfrak{F}$ a *generating family* for the structure. In particular the family of all closed subsets of S with respect to a topology τ in S generates a Borel structure which we shall call the *Borel structure generated by the topology*. The corresponding Borel space will be said to be that *associated with* or *defined by* the given topological space. If S is a Borel space and $\mathfrak{F}$ is a family of Borel subsets of S we shall say that $\mathfrak{F}$ *separates* S or is a *separating family* if given any two points p and q in S with $p\neq q$ there exists $E\in\mathfrak{F}$ with $p\in E$ and $q\notin E$. If a separating family exists we shall say that S is *separated*.

Let S_1 and S_2 be Borel spaces and let f be a function defined on S_1 and having values in S_2. If $f^{-1}(E)$ is a Borel subset of S_1 whenever E is a Borel subset of S_2 we shall say that f is a *Borel function*. If f is one-to-one and onto and if f and f^{-1} are both Borel functions we shall say that f is a *Borel isomorphism* of S_1 onto S_2 and that S_1 and S_2 are isomorphic as Borel spaces. Evidently if S_1 and S_2 have Borel structures generated by topologies with respect to which a function f is a homeomorphism then f is a Borel isomorphism. The converse of course is not true.

If S is a Borel space and E is an arbitrary subset of S then E becomes a Borel space on defining the Borel subsets of E to be the sets $E\cap F$ where F runs over the Borel subsets of S. We shall call the Borel space E a *subspace* of the Borel space S. If the Borel structure in S is generated by a topology we may also obtain a Borel structure in E by taking the Borel structure generated by the restriction to E of the topology in S. It is a useful fact that these two Borel structures are always the same. Indeed it is obvious that every open subset of E is a Borel set in the first sense and hence that every Borel set in the second sense is also one in the first. To prove the converse consider the set $\mathcal{E}$ of all subsets of S which intersect E in a Borel set in the second sense.

Clearly $\mathcal{E}$ is a Borel structure for S which contains all open subsets of S. Thus $\mathcal{E}$ includes all Borel subsets of S. Thus every Borel subset of E in the first sense is also one in the second sense.

If $\{S_\alpha\}$ is any family of Borel spaces such that $S_\alpha \cap S_\beta = 0$ for all α and β with $\alpha \neq \beta$ we define a Borel structure in $S = \bigcup_\alpha S_\alpha$ by defining a subset E of S to be a Borel set if and only if $S_\alpha \cap E$ is a Borel set for all α. We shall call S the *union* of the Borel spaces S_α. We make the Cartesian product $\prod_\alpha S_\alpha$ of a family of Borel spaces into a Borel space by fitting it with the Borel structure generated by the "elementary rectangles." Here by an elementary rectangle we mean a set defined in the following way. Choose an α and a Borel set $F_\alpha \subseteq S_\alpha$ and form the set of all $x = \{x_\beta\} \in \prod_\beta S_\beta$ such that $x_\alpha \in F_\alpha$. Clearly the Borel structure generated by the topology of the Cartesian product of a family of topological spaces is identical with the Cartesian product of the Borel structures generated by the topologies in the factors.

Let S be a Borel space and let r be an equivalence relation in S. Let $\tilde{S}$ denote the set of all equivalence classes and let $r(x)$ for each $x \in S$ denote the equivalence class to which x belongs. Let $\mathfrak{F}$ be the set of all $E \subseteq \tilde{S}$ such that $r^{-1}(E)$ is a Borel set. Obviously $\mathfrak{F}$ is a Borel structure for $\tilde{S}$ and is the largest such that $x \to r(x)$ is a Borel function. $\tilde{S}$ equipped with the Borel structure $\mathfrak{F}$ we shall call the *quotient Borel space* of S by the equivalence relation r.

2. **Countably separated and countably generated Borel spaces.** Let S be a Borel space. If S has a countable separating family of Borel sets we shall say that S is *countably separated.* If S has a countable generating family of Borel sets and is separated we shall say that it is *countably generated.* Clearly every countably generated Borel space is countably separated and every subspace of a countably generated (respectively countably separated) Borel space is countably generated (respectively countably separated).

THEOREM 2.1. *Let M be the topological product of denumerably many replicas of the discrete two element space. Then a necessary and sufficient condition that a Borel space S be countably generated is that it be isomorphic to a subspace of the Borel space associated with M.*

Proof. Since M is separable the sufficiency is obvious. To prove the necessity let $E_1, E_2, \cdots$ be a countable generating family for S and let ϕ_j be the characteristic function of E_j. Let $\phi(x)$ denote the infinite sequence $\phi_1(x), \phi_2(x), \cdots$. Then ϕ is one-to-one and maps S onto a subset of M. Let F_j be the set of all points of M with jth term equal to one. Clearly the F_j generate the Borel subsets of M. Hence the $F_j \cap \phi(S)$ generate the Borel sets in $\phi(S)$. But $\phi(E_j) = F_j \cap \phi(S)$. Hence ϕ is an isomorphism and the theorem is proved.

THEOREM 2.2. *Let f be a Borel function whose domain S_1 and range S_2 are both countably generated Borel spaces. Then there exist isomorphisms ϕ and ϕ'*

of S_1 and S_2 into the M of Theorem 2.1 *such that $\phi' f \phi^{-1}$ is the restriction to $\phi(S_1)$ of a continuous mapping of M into M.*

Proof. Let $E_1, E_2, \cdots$ be a countable generating family for S_2 and let $F_1, F_2, \cdots$ be a countable generating family for S_1 such that $f^{-1}(E_j) = F_{2j}$ for all j. Let ϕ_j be the characteristic function of F_j and let ϕ_j' be the characteristic function of E_j. Let $\phi(x) = \phi_1(x), \phi_2(x), \cdots$ and let $\phi'(x) = \phi_1'(x), \phi_2'(x), \cdots$. ϕ and ϕ' are then Borel isomorphisms of S_1 and S_2 respectively into M. For each $\lambda = \lambda_1, \lambda_2, \cdots \in M$ let $\theta(\lambda) = \lambda_2, \lambda_4, \lambda_6, \cdots$. Then θ is clearly a continuous mapping of M onto M. Moreover the jth component of $\theta(\phi(x))$ is equal to one if and only if the $2j$th component of $\phi(x)$ is equal to one; that is if and only if $x \in F_{2j}$. But $x \in F_{2j}$ if and only if $f(x) \in E_j$; that is, if and only if the jth component of $\phi'(f(x))$ is equal to one. Hence $\phi' f = \theta\phi$ and the truth of the theorem follows.

3. **Standard Borel spaces.** Let M_1 and M_2 be complete separable metric spaces and let E_1 and E_2 be noncountable Borel subsets of M_1 and M_2 respectively. It follows immediately from Remark 1 on p. 358 of [6](2) that E_1 and E_2 are isomorphic as Borel spaces and hence that there is a tremendous array of different mathematical objects with identical Borel structures. We call a Borel space with this structure a standard Borel space. More generally we define a Borel space to be *standard* whenever it is Borel isomorphic to the Borel space associated with a Borel subset of a complete separable metric space. It follows from what has been said that two standard Borel spaces are Borel isomorphic if and only if they have the same cardinal number and that the only infinite cardinals possible are $\aleph_0$ and $2^{\aleph_0}$. Since noncountable standard Borel spaces are mutually Borel isomorphic and since the Borel space associated with the M of Theorem 2.1 is standard one can give a more intrinsic definition as follows. The Borel space S is standard if it is separated and it has at most countably many elements or has noncountably many elements and admits a countable separating family of Borel sets $E_1, E_2, \cdots$ such that for each subset J of the positive integers there exists one and only one $x \in S$ with the property that $x \in E_j$ if and only if $j \in J$.

THEOREM 3.1. *If $S_1, S_2, \cdots$ are standard Borel spaces then their union and their Cartesian product are standard Borel spaces.*

Proof. Let R be the Borel space associated with the real line. Let I_j be the interval $j-1 < x < j$. Then S_j is Borel isomorphic to the subspace of I_j defined by a Borel subset K_j of I_j. Hence the union of the S_j is Borel isomorphic to the subspace defined by the Borel subset $\bigcup_{j=1}^{\infty} K_j$ of R and so is standard. Clearly there exists a compact separable metric space containing $\aleph$ elements where $\aleph = 2^{\aleph_0}$, $\aleph_0$ or is finite. Hence S_j is isomorphic to the Borel

(2) I am indebted to J. C. Oxtoby for this reference. I had originally deduced the fact in question from two other theorems in the first edition of Kuratowski's book.

space associated with a compact separable space M_j. Hence $\prod_{j=1}^{\infty} S_j$ is isomorphic to the Borel space associated with the compact separable space $\prod_{j=1}^{\infty} M_j$ and so is standard.

The following consequence of a deep theorem of Souslin will be of considerable importance for us.

THEOREM 3.2. *Let f be a one-to-one Borel function whose domain is a standard Borel space and whose range is contained in a countably generated Borel S space. Then the range of f is a Borel subset of S and f is a Borel isomorphism of its domain with its range. In particular the range of f is standard.*

Proof. There is no loss in generality in supposing that the domain of f is a separable complete metric space M_1. Moreover by Theorem 2.1 we may assume in addition that the range of f is a subset of a separable complete metric space M_2. Then as is shown by Kuratowski on p. 396 of [6] it follows from a theorem of Souslin that for every Borel subset E of $M_1, f(E)$ is a Borel subset of M_2 and hence of $f(M_1)$. Thus f is a Borel isomorphism and $f(M_1)$ must be standard.

COROLLARY 1. *A necessary and sufficient condition that a subspace of a standard Borel space be standard is that it be defined by a Borel subset.*

COROLLARY 2. *There exist countably generated Borel spaces which are not standard.*

COROLLARY 3. *A subset of a countably generated Borel space which is standard as a Borel subspace is a Borel subset.*

THEOREM 3.3. *Let S be a standard Borel space and let $E_1, E_2, \cdots$ be any countable separating family of Borel subsets of S. Then $E_1, E_2, \cdots$ is a generating family for the Borel structure of S.*

Proof (3). Let ϕ be the identity map from S with the given standard Borel structure to S with the Borel structure generated by the E_j. By Theorem 3.2, ϕ is a Borel isomorphism. Hence the E_j generate the given structure.

COROLLARY. *Let $E_1, E_2, \cdots$ be a countable separating family of subsets of a set S. Then there is at most one way of introducing a standard Borel structure in S so that the E_j are all Borel sets.*

For many mathematical objects there exists an obvious family of subsets deserving to be included in any "reasonable" Borel structure and having a countable separating subfamily. If such an object has a "reasonable" Borel structure which is also standard the preceding corollary shows that it is unique.

(3) I am indebted to R. S. Palais for suggesting this proof. My original proof was more complicated. I am also indebted to Dr. Palais for pointing out some errors and obscurities in the first few sections of the paper.

4. **Analytic Borel spaces.** Let S be a countably generated Borel space which is a Borel image of a standard Borel space; that is, is the range of a Borel function whose domain is a standard Borel space. If this function is one-to-one then as we have seen S itself is standard. When the function is not one-to-one, S need not be standard but as we shall see has many of the properties of standard spaces. We shall call a countably generated Borel image of a standard Borel space an *analytic* Borel space. The reason for this terminology will be made clear by the next theorem. We shall encounter analytic Borel spaces chiefly as quotient spaces of standard Borel spaces.

THEOREM 4.1. *A necessary and sufficient condition that a Borel space be analytic is that it be Borel isomorphic to the Borel space associated with an analytic subset of the real line.*

Proof. Since an analytic subset of the real line may be defined as the image of a Borel set in the real line by a continuous real valued function of a real variable the sufficiency of the condition is obvious. To prove the necessity let f be a Borel function whose domain is a standard Borel space S_1 and whose range is the given analytic Borel space S_2. By Theorem 2.2, S_1 and S_2 may be regarded as the Borel spaces defined by subsets of totally disconnected compact spaces M_1 and M_2 in such a way that f is the restriction to S_1 of a continuous function defined on M_1. S_1 will be a Borel subset of M_1 by Theorem 3.2, Corollary 1. Since M_1 and M_2 are totally disconnected and compact they may be imbedded homeomorphically in the real line and the result follows.

We have the following analogue of Theorem 3.2.

THEOREM 4.2. *Let f be a one-to-one Borel function whose domain is an analytic Borel space and whose range is a countably generated Borel space. Then the range of f is also analytic and f is a Borel isomorphism.*

Proof. That the range of f is analytic follows from the definition of analyticity and that f is a Borel isomorphism is an immediate consequence of Théorème 3 on p. 398 of [6].

The next theorem is a generalization of Theorem 3.3 and is deduced from Theorem 4.2 just as Theorem 3.3 was deduced from Theorem 3.2. We leave details to the reader.

THEOREM 4.3. *Let S be an analytic Borel space and let $E_1, E_2, \cdots$ be any countable separating family of Borel subsets of S. Then $E_1, E_2, \cdots$ is a generating family for the Borel structure of S.*

COROLLARY. *Let $E_1, E_2, \cdots$ be a countable separating family of subsets of a set S. Then there is at most one way of introducing an analytic Borel structure in S so that the E_j are all Borel sets.*

5. **Quotient spaces of analytic Borel spaces.** Many Borel spaces appear as quotient spaces of quite regular Borel spaces. In a sense made precise by

the following theorem such quotients are either quite regular or quite irregular.

THEOREM 5.1. *Let S be a countably separated Borel space such that $S=f(S_0)$ for some analytic Borel space S_0 and some Borel function f. Then S is analytic.*

Proof. Let $\mathcal{B}$ be the given Borel structure in S and let $\mathcal{B}'$ be the Borel structure generated by a countable separating family of sets in $\mathcal{B}$. S, $\mathcal{B}'$ being countably generated and a Borel image of S_0 is then analytic. If $\mathcal{B}\neq\mathcal{B}'$ let E be a member of $\mathcal{B}-\mathcal{B}'$. Let $\mathcal{B}''$ be the Borel structure generated by $\mathcal{B}'$ and E. By the argument just given, S, $\mathcal{B}''$ is analytic. By the corollary to Theorem 4.3, $\mathcal{B}'=\mathcal{B}''$. Hence $\mathcal{B}=\mathcal{B}'$. Hence S is countably generated. Hence S is analytic.

COROLLARY. *If a quotient space of an analytic Borel space is countably separated then it is analytic.*

It is obvious that $\tilde{S}$ is countably separated if and only if there exist countably many real valued Borel functions on S each of which is a constant on the equivalence classes and no two of which agree on all equivalence classes. In other words $\tilde{S}$ is countably separated if and only if there exists a countable complete set of "invariants" for the equivalence relation r. In still other words $\tilde{S}$ is countably separated if and only if the equivalence classes can be "classified" by means of a "suitable" system of invarints.

We shall show later that the coset space $\mathcal{G}/G$ where $\mathcal{G}$ is a separable locally compact group is countably separated if and only if G is closed. Thus we get a simple example of a noncountably separated $\tilde{S}$ if we let S be the real line and let x and y be equivalent if and only if $x-y$ is rational.

One also "classifies" equivalence classes by finding "canonical forms." This suggests that the countable separatedness of $\tilde{S}$ might be closely related to the existence of a Borel subset of S having exactly one element in common with each equivalence class. It turns out indeed that in the special case in which S is standard and equivalence is equivalence under a suitable transformation group then the existence of a Borel "cross section" for r does imply that $\tilde{S}$ is countably separated. Let us define a *Borel group* to be a group $\mathcal{G}$ with a Borel structure $\mathcal{B}$ such that $x, y\rightarrow xy^{-1}$ is a Borel function from $\mathcal{G}\times\mathcal{G}$ to $\mathcal{G}$. According as the underlying Borel structure is standard, analytic etc., we shall say that $\mathcal{G}$ is a *standard Borel group*, an *analytic Borel group*, etc. Let S be a Borel space and let $\mathcal{G}$ be a Borel group. Let h be a Borel function from $S\times\mathcal{G}$ to S such that (a) For each fixed $y\in\mathcal{G}$, $x\rightarrow h(x, y)$ is a Borel isomorphism T_y of S with itself and (b) $y\rightarrow T_y$ is a homomorphism of $\mathcal{G}$ into the group of all Borel isomorphisms of S with itself. We shall call the system S, $\mathcal{G}$, h a *Borel transformation group*.

THEOREM 5.2. *Let S, $\mathcal{G}$, h be a Borel transformation group such that S is standard and $\mathcal{G}$ is analytic. Let $T_y(x)=h(x, y)$ and let r be the equivalence rela-*

tion such that $r(x_1)=r(x_2)$ if and only if there exists y in $\mathcal{G}$ such that $T_y(x_1)=x_2$. Then $\tilde{S}$ will be standard whenever there exists a Borel subset E of S which has one and only one element in common with each equivalence class.

Proof. Let F be a Borel subset of E. It will suffice to show that $r(F)$ is a Borel set since r is one-to-one from E onto $\tilde{S}$ and r is a Borel function. But $r(F)$ is a Borel set if and only if $r^{-1}(r(F))$ is a Borel set. Moreover $r^{-1}(r(F)) = h(F\times\mathcal{G})$. Hence we need only show that $h(F\times\mathcal{G})$ is a Borel set. Since S is standard we may assume without loss of generality that S is the real line. But then $h(F\times\mathcal{G})$ is clearly an analytic subset of the real line. The same argument shows that $h((E-F)\times\mathcal{G})$ is an analytic subset of the real line. Since $h((E-F)\times\mathcal{G})$ and $h(F\times\mathcal{G})$ are complementary subsets of S it follows from Corollary 1 on p. 395 of [6] that they are both Borel sets and the proof is complete.

6. **Measures in Borel spaces.** Let S be a Borel space. By a *Borel measure* in (or on) S we shall mean a function μ from the Borel subsets of S to the non-negative real numbers and ∞ such that $\mu(\bigcup_{n=1}^{\infty} E_n) = \sum_{n=1}^{\infty} \mu(E_n)$ whenever the E_n are Borel sets and $E_n \cap E_m = 0$ for $n\neq m$. We shall deal only with those Borel measures which are *σ-finite* in the sense that S is a union of countably many Borel sets of finite measure. If $\mu(S)<\infty$ we shall say that μ is a finite Borel measure. Let μ be a σ-finite Borel measure in S. We shall denote by $\mathfrak{N}_\mu$ the set of all sets $E\subseteq S$ such that $\mu(E')=0$ for some Borel set $E'\supseteq E$ and call its members the null sets of S (with respect to μ). Let $\mathfrak{M}_\mu$ denote the smallest Borel structure in S which includes $\mathfrak{N}_\mu$ and the given Borel structure in S. The members of $\mathfrak{M}_\mu$ we shall call the *μ measurable* subsets of S. It is easy to see that the members of $\mathfrak{M}_\mu$ are just the sets $(E-N_1)\cup N_2$ where E is a Borel set and N_1 and N_2 are in $\mathfrak{N}_\mu$ and that there is a unique σ-finite Borel measure $\bar{\mu}$ in S, such that $\mu(E)=\bar{\mu}(E)$ for all Borel sets E in S. We shall call $\bar{\mu}$ the *completion* of μ. We shall say that the σ-finite measure μ is *standard* if there exists a Borel subset E of S such that E regarded as a subspace is standard and such that $\mu(S-E)=0$.

THEOREM 6.1. *Every σ-finite Borel measure in an analytic Borel space is standard.*

Proof. Evidently there is no loss in generality in assuming that the given Borel space is the subspace of the real line R defined by some analytic subset A of R. Since for every σ finite Borel measure there is a finite Borel measure with the same sets of measure zero we may suppose in addition that our given measure μ is finite. For each Borel subset E of R let $\nu(E)=\mu(E\cap A)$. Then ν is a finite Borel measure in R. Hence (cf. [6, p. 391]) every analytic subset of R is ν measurable. Hence there exists a Borel subset F of R such that $F\subseteq A$ and $\nu(A-F)=\mu(A-F)=0$.

It is natural to define a Borel space to be *metrically standard* if every σ-finite measure in it is standard. Theorem 6.1 then asserts that an analytic

Borel space is always metrically standard. Analogously we define a σ-finite measure μ to be countably separated (respectively countably generated) if there exists a Borel subset E of S such that $\mu(S-E)=0$ and E regarded as a subspace is countably separated (respectively countably generated). If every σ-finite measure in S is countably separated (respectively countably generated) we say that S is *metrically countably separated* (respectively *metrically countably generated*). If μ is a finite measure in the Borel space S and $\tilde{S}$ is the quotient space defined by an equivalence relation r then by $\tilde{\mu}$ the *image of μ in $\tilde{S}$* we mean the Borel measure in $\tilde{S}$ such that $\tilde{\mu}(E)=\mu(r^{-1}(E))$ for all Borel subsets E of $\tilde{S}$.

THEOREM 6.2. *Let μ be a finite standard measure in the Borel space S. Let $\tilde{S}$ be a quotient space of S and let $\tilde{\mu}$ be the image of μ in S. Then if $\tilde{\mu}$ is countably separated it is standard.*

Proof. Let r be the natural map of S on $\tilde{S}$ and let E be a Borel subset of $\tilde{S}$ such that E defines a countably separated subspace and $\tilde{\mu}(\tilde{S}-E)=0$. Then $\mu(S-r^{-1}(E))=0$. Let F be a Borel subset of S such that $\mu(S-F)=0$ and F defines a standard Borel subspace of S. Then $\mu(S-(F\cap r^{-1}(E))=0$ and $F\cap r^{-1}(E)$ being a Borel subset of F defines a standard Borel subspace of S. Since $r(F\cap r^{-1}(E))\subseteq E$ it defines a countably separated subspace of S. By Theorem 5.1 this subspace is analytic. The truth of the theorem now follows at once from the fact that $\tilde{\mu}(E-(F\cap r^{-1}(E)))=0$ and Theorem 6.1.

COROLLARY. *If S is metrically standard and every finite measure in $\tilde{S}$ is of the form $\tilde{\mu}$ then $\tilde{S}$ is metrically standard if it is metrically countably separated.*

THEOREM 6.3. *Let S_1 and S_2 be Borel spaces and let S_2 be standard. Let μ be a standard measure in S_1. Let A be a Borel subset of $S_1\times S_2$ such that for each $x\in S_1$ there exists $y\in S_2$ so that $x, y\in A$. Then there exists a Borel subset N of S_1 and a Borel function ϕ from S_1-N to S_2 such that $x, \phi(x)\in A$ for all $x\in S_1-N$ and $\mu(N)=0$.*

Proof. This theorem is a reformulation of a lemma of von Neumann [14, Lemma 5, p. 448]. The deduction of our version proceeds as follows. By the definition of standard measure there is clearly no loss in generality in assuming that S_1 is also standard. Since the theorem is trivial if either S_1 or S_2 is countable it follows from the remark at the beginning of §3 that we need only consider the case in which S_1 and S_2 are both the closed unit interval. Let f be the continuous function on A which takes x, y into x. By the lemma of von Neumann there exists a function g from S_1 to A such that $g^{-1}(\mathcal{O})$ is μ measurable for every open subset of $S_1\times S_2$ and such that $f(g(x))=x$ for all $x\in S_1$. Thus $g(x)=x, \psi(x)$ where $\psi(x)\in S_2$. If $\mathcal{O}$ is an open set in S_2 then $g^{-1}(S_1\times\mathcal{O})=\psi^{-1}(\mathcal{O})$. Hence ψ is μ measurable. Hence there exists a Borel function ϕ from S_1 to S_2 and a Borel set N of μ measure zero such that $\phi(x)=\psi(x)$ for all $x\in S_1-N$. Since $x, \psi(x)\in A$ for all $x\in S$ it follows that x,

$\phi(x) \in A$ for all $x \in S_1 - N$ and the proof is complete.

If S is any Borel space and μ is a σ-finite Borel measure in S we shall denote by C_μ the class of all σ finite Borel measures in S having the same null sets as μ. A family C of measures in S will be said to constitute a *measure class* in S if it is of the form C_μ for some nontrivial σ-finite Borel measure μ in S. It is suggestive to think of a measure class as a sort of generalized subset which is described not by saying which points are in it but by saying which sets of points (the sets of measure zero) are disjoint from it. When there is a largest set of measure zero (e.g. when the complement of some countable set is of measure zero) then the complement of this largest set of measure zero completely determines the measure class. In this case the "generalized subset" associated with C may be identified with an ordinary subset. In particular if S is countable all "generalized subsets" are ordinary. When S is not countable the measure classes often take over the rôle of subsets; e.g. in the unitary equivalence theory of self-adjoint operators in Hilbert space. In the same spirit a function defined only up to a set of measure zero may be regarded as a function whose domain of definition is the generalized subset associated with the appropriate measure class. If T is a Borel isomorphism of S with itself then the set of all measures $E \rightarrow \mu(T^{-1}(E))$ where μ is in the measure class C is again a measure class. We call this measure class the transform of C by T and denote it by $T(C)$.

7. **Invariant measure classes in Borel groups.** Let $\mathcal{G}$ be a Borel group and let C be a measure class in $\mathcal{G}$. We shall say that C is right (left) invariant if $R_x(C) = C$ ($L_x(C) = C$) for all $x \in \mathcal{G}$ where R_x and L_x are the Borel automorphisms $y \rightarrow yx$ and $y \rightarrow xy$ respectively. If $\mathcal{G}$ is the Borel group associated with a separable locally compact topological group then all left and right invariant Haar measures are contained in a common measure class which is both right and left invariant. Moreover, as is shown in Lemma 3.3 of [8] this is the only measure class in $\mathcal{G}$ which is either left or right invariant. We shall now give a converse of this result. First we prove some preliminary lemmas.

LEMMA 7.1. *Let μ be a finite Borel measure in the Borel group $\mathcal{G}$. Then, for each Borel set E in $\mathcal{G}$, $\mu(sE)$ and $\mu(Es)$ are μ measurable functions of s.*

Proof. Since $\mathcal{G}$ is a Borel group, xy^{-1} is a Borel function on $\mathcal{G} \times \mathcal{G}$. Hence $x \rightarrow x^{-1}$ is a Borel function and $x, y \rightarrow x^{-1}y$ and $x, y \rightarrow xy$ are both Borel functions. Hence, if $T(s, x) = s, sx$, T is a Borel isomorphism of $\mathcal{G} \times \mathcal{G}$ with itself. Hence $T(\mathcal{G} \times E)$ is a Borel set for all Borel sets E in $\mathcal{G}$. But $T(\mathcal{G} \times E) \cap (s \times \mathcal{G}) = s \times sE$. Hence applying the Fubini theorem to $\mu \times \mu$ and the characteristic function ψ of $T(\mathcal{G} \times E)$ we see that $\mu(sE) = \int \psi(s, x) d\mu(x)$ is measurable in s. The proof for $\mu(Es)$ is analogous.

It is clear that if one member of a measure class is standard, countably generated or countably separated then so are all of the others. Hence we may speak without ambiguity of measure classes having these various properties.

LEMMA 7.2. *Let the Borel group $\mathcal{G}$ have a left (right) invariant measure class C. Then $\mathcal{G}$ has a measure class which is both right and left invariant.*

Proof. Let μ' be a finite member of the left invariant measure class C. Let $\mu(E) = \int_{\mathcal{G}} \mu'(Ex)d\mu'(x)$. Then μ is a finite Borel measure on $\mathcal{G}$. Moreover $\mu(E)=0$ if and only if $\mu'(Ex)=0$ for μ' almost all x. Hence $\mu(sE)=0$ if and only if $\mu'(sEx)=0$ for μ' almost all x. But $\mu'(Ex)=0$ if and only if $\mu'(sEx)=0$ since $C_{\mu'}$ is left invariant. Hence C_μ is also left invariant. On the other hand $\mu(Es)=0$ if and only if $\mu'(Esx)=0$ for μ' almost all x. But $\mu'(Esx)=0$ for μ' almost all x if and only if $\mu'(Ex)=0$ for μ' almost all x; that is, if and only if $\mu(E)=0$. Thus C_μ is also right invariant. Since

$$\mu(\mathcal{G}) = \int_{\mathcal{G}} \mu'(\mathcal{G}x)d\mu'(x) = (\mu'(\mathcal{G}))^2 \neq 0,$$

μ is nontrivial. If C is right invariant then C_ν is left invariant where $\nu(E) = \mu'(E^{-1})$ and the above argument applies.

LEMMA 7.3. *Let C be a countably generated measure class in the Borel group $\mathcal{G}$ which is both left and right invariant. Then C contains a left invariant measure and a right invariant measure.*

Proof. It will clearly suffice to prove the existence of a left invariant measure. Let μ be a finite member of C. For each $s \in \mathcal{G}$ let $\mu_s(E) = \mu(sE)$ for all Borel sets $E \subseteq \mathcal{G}$. Then μ_s and μ have the same null sets. Hence by the Radon-Nikodym theorem there exists a Borel function ρ_s such that $\mu_s(E) = \int_E \rho_s(x)d\mu(x)$ for all Borel sets E. Moreover we may choose ρ_s so as to be everywhere positive. Since $\mu_s(E)$ is measurable in s we may apply Lemma 3.1 of [8] to deduce the existence of a $\mu \times \mu$ measurable function ρ' on $\mathcal{G} \times \mathcal{G}$ such that for all s in $\mathcal{G}$, $\rho'(s, x) = \rho_s(x)$ for almost all x in $\mathcal{G}$. We remark that the hypothesis in Lemma 3.1 of [8] according to which ρ_s must be bounded is needlessly strong. The proof remains valid when we know only the existence of the integrals $\int_E \rho_s(x)d\mu(x)$. We wish to find a positive Borel function λ such that if we set $\nu_\lambda(E) = \int_E \lambda(x)d\mu(x)$ for all Borel sets E the resulting member of C will be left invariant. Now

$$\nu_\lambda(sE) = \int_{sE} \lambda(x)d\mu(x) = \int_E \lambda(sx)d\mu_s(x) = \int_E \lambda(sx)\rho'(s, x)d\mu(x).$$

Thus $\nu_\lambda(sE) = \nu_\lambda(E)$ for all E if and only if $\lambda(x) = \lambda(sx)\rho'(s, x)$ for μ almost all x. In other words it will suffice to show that there exists a positive Borel function λ such that for all s we have $\rho'(s, x) = \lambda(x)/\lambda(sx)$ for μ almost all x. Consider μ_{st}. $\mu_{st}(E) = \int_E \rho'(st, x)d\mu(x)$ and $\mu_{st}(E) = \mu(stE) = \mu_s(tE) = \int_{tE} \rho'(s, x)d\mu(x) = \int_E \rho'(s, tx)\rho'(t, x)d\mu(x)$ for all Borel sets E. Hence for all s and t we have $\rho'(st, x) = \rho'(s, tx)\rho'(t, x)$ for almost all x. The argument of Lemma 3.2 of [8] modified in a slight and obvious manner shows that $\rho'(s, tx)$ and $\rho'(st, x)$ are

$\mu\times\mu\times\mu$ measurable on $\mathcal{G}\times\mathcal{G}\times\mathcal{G}$. Hence by the Fubini theorem there exists $x_0\in\mathcal{G}$ such that

$$\rho'(st, x_0) = \rho'(s, tx_0)\rho'(t, x_0)$$

for $\mu\times\mu$ almost all pairs s, t. Hence

$$\rho'(s, tx_0) = \rho'(st, x_0)/\rho'(t, x_0)$$

for $\mu\times\mu$ almost all pairs s, t. If we set $x=tx_0$ it follows that for μ almost all s we have $\rho'(s, x)=\rho'(sxx_0^{-1}, x_0)/\rho'(xx_0^{-1}, x_0)$ for μ almost all x. Thus if we choose $\lambda(x)=1/\rho'(xx_0^{-1}, x_0)$ we see that for μ almost all s, $\rho'(s, x)=\lambda(x)/\lambda(sx)$ for μ almost all x and hence that for all E $\nu_\lambda(sE)=\nu_\lambda(E)$ for μ almost all s. Let $\mathcal{G}_0$ be the set of all $s\in\mathcal{G}$ such that for all E $\nu_\lambda(sE)=\nu_\lambda(E)$. Then $\mathcal{G}_0$ is a subgroup and $\mu(\mathcal{G}-\mathcal{G}_0)=0$. But if $s\notin\mathcal{G}_0$ then $s\mathcal{G}_0\subseteq\mathcal{G}-\mathcal{G}_0$. Hence $\mu(s\mathcal{G}_0)=0$. Hence $\mu(\mathcal{G}_0)=0$ and this is a contradiction. Thus $\mathcal{G}_0=\mathcal{G}$ so ν_λ is left invariant as was to be proved.

LEMMA 7.4. *Let ν be a σ-finite left invariant countably separated Borel measure in the Borel group $\mathcal{G}$. Then for each $s\in\mathcal{G}$ with $s\neq e$ there exists a Borel set $E\subseteq\mathcal{G}$ such that $\nu(E\cap sE)<\nu(E)<\infty$.*

Proof. Suppose the contrary. Then for some $s\neq e$, $\nu(E\cap sE)=\nu(E)$ for every Borel set with $\nu(E)<\infty$. Let $E_0, E_1, \cdots$ be a family of Borel sets such that $\nu(E_0)=0$, $E_1, E_2, \cdots$ is a separating family for the Borel subsets of $\mathcal{G}-E_0$ and $\nu(E_j)<\infty$ for all j. Let $N=E_0\cup\bigcup_{n=1}^{\infty}((E_j-sE_j)\cup(sE_j-E_j))$. Then $\nu(N)=0$. Let $N'=\bigcup_{n=-\infty}^{\infty}s^jN$. Then $\nu(N')=0$ and $sN'=N'$. Let $F_j=E_j-E_j\cap N'$. Then $sF_j=F_j$ for all j. But every point of $\mathcal{G}-E_0$ is the intersection of the E_j which contain it. Hence every point x_0 of $\mathcal{G}-N'$ is the intersection of the F_j which contain it. Hence $sx_0=x_0$. Hence $s=e$ and we have a contradiction which proves the lemma.

THEOREM 7.1. *Let $\mathcal{G}$ be an analytic Borel group and let there exist in $\mathcal{G}$ either a left or a right invariant measure class. Then there exists in $\mathcal{G}$ a unique locally compact topology whose Borel sets are the given ones and under which $\mathcal{G}$ is a topological group.*

Proof. By Lemma 7.2 $\mathcal{G}$ has a two sided invariant measure class C and by Lemma 7.3 C contains a left invariant measure ν. We now apply the theorem proved by Weil in appendix I of [15]. By hypothesis $x, y\rightarrow xy^{-1}$ is a Borel function from $\mathcal{G}\times\mathcal{G}$ to $\mathcal{G}$. Hence $x, y\rightarrow y^{-1}x, y=T(x, y)$ is a Borel function and T^{-1}, since $T^{-1}(x, y)=yx, y$, is also a Borel function. Hence Weil's hypothesis M is satisfied for $\mathcal{G}$ and ν. By Lemma 7.4 Weil's hypothesis M' is also satisfied. Hence (p. 144 of Weil's appendix) the sets $E\cdot E^{-1}$ where E ranges over the Borel sets of finite measure form a defining system of neighborhoods of the identity for a topology τ in $\mathcal{G}$ under which $\mathcal{G}$ is a dense subgroup of a locally compact topological group $\bar{\mathcal{G}}$. Moreover (again as shown by Weil) every continuous function with compact support on $\bar{\mathcal{G}}$ is measurable on $\mathcal{G}$ and its Haar integral on $\bar{\mathcal{G}}$ is equal to the ν integral of its restriction to $\mathcal{G}$. We shall now show that in the case at hand $\mathcal{G}$ and $\bar{\mathcal{G}}$ actually coincide. Let ν' be the Haar measure in $\bar{\mathcal{G}}$ which extends ν in the sense just described. The continuous functions with compact support form a dense subspace of $\mathcal{L}^2(\bar{\mathcal{G}}, \nu')$ and this subspace is unitarily equivalent to a subspace of $\mathcal{L}^2(\mathcal{G}, \nu)$. Since $\mathcal{G}$ is analytic and hence countably generated $\mathcal{L}^2(\mathcal{G}, \nu)$ is separable. Hence $\mathcal{L}^2(\bar{\mathcal{G}}, \nu')$ is separable. Hence, as one can easily show, $\bar{\mathcal{G}}$ is separable. Now let f and g be arbitrary continuous functions with compact support defined on $\bar{\mathcal{G}}$. Since their restrictions to $\mathcal{G}$ are ν measurable there exist Borel functions on $\mathcal{G}$ f' and g' which differ from the respective restrictions of f and g only on null sets. Thus for all $x \in \mathcal{G}$,

$$\int_{\bar{\mathcal{G}}} f(yx)g(y)d\nu'(y) = \int_{\mathcal{G}} f'(yx)g'(y)d\nu(y).$$

Since $f'(yx)g'(y)$ is a Borel function on $\mathcal{G} \times \mathcal{G}$ it follows from the Fubini theorem that h is a Borel function on $\mathcal{G}$ where $h(x) = \int_{\mathcal{G}} f'(yx)g'(y)d\nu(y)$. But h is the restriction to $\mathcal{G}$ of a continuous function on $\bar{\mathcal{G}}$. Thus there exists a sequence $h_1, h_2, \cdots$ of continuous functions on $\bar{\mathcal{G}}$ which separate points and are Borel functions when restricted to $\mathcal{G}$. Hence there exists a countable separating family of Borel sets in $\bar{\mathcal{G}}$ whose intersections with $\mathcal{G}$ are Borel sets in $\mathcal{G}$. Since every countable separating family in $\bar{\mathcal{G}}$ is a generating family (Theorem 3.3) it follows that the intersection with $\mathcal{G}$ of every Borel set in $\bar{\mathcal{G}}$ is a Borel subset of $\mathcal{G}$. Hence the identity mapping of $\mathcal{G}$ in $\bar{\mathcal{G}}$ is Borel. Hence by Theorem 4.2 this mapping is a Borel isomorphism. Let $\mathcal{S}$ be a standard subset of $\mathcal{G}$ such that $\nu(\mathcal{G} - \mathcal{S}) = 0$. By Theorem 3.2 $\mathcal{S}$ is a Borel subset of $\bar{\mathcal{G}}$. Since $\int_{\bar{\mathcal{G}}} f(x)d\nu'(x) = \int_{\mathcal{G}} f(x)d\nu(x) = \int_{\mathcal{S}} f(x)d\nu(x)$ for all continuous f on $\bar{\mathcal{G}}$ with compact support it follows that $\nu'(\bar{\mathcal{G}} - \mathcal{S}) = 0$. Hence $\nu'(\bar{\mathcal{G}} - \mathcal{G}) = 0$ and $\mathcal{G}$ is ν' measurable. Hence $\nu'(\mathcal{G}) = \nu'(\bar{\mathcal{G}}) \neq 0$. Hence $\nu'(x\mathcal{G}) \neq 0$ for any x in $\bar{\mathcal{G}}$. Since $x\mathcal{G} = \mathcal{G}$ or is disjoint from $\mathcal{G}$ it follows that $x\mathcal{G} = \mathcal{G}$ for all x. Hence $\mathcal{G} = \bar{\mathcal{G}}$. It remains to prove the uniqueness of the locally compact topology for $\mathcal{G}$. Now if two locally compact topologies for the same group have the same Borel sets then the same unitary representations are continuous with respect to both. Indeed, as is well known, a unitary representation U of a locally compact group is continuous if and only if the functions $x \to (U_x(\phi), \psi)$ are Borel functions for all ϕ and ψ in the space of U. Thus the two topologies have a separating family of continuous functions in common. Hence the two topologies have the same compact sets and restricted to compact sets are identical. Since both topologies are locally compact the identity mapping is bicontinuous at every point and hence is a homeomorphism.

COROLLARY. *If an analytic Borel group has a measure class which is either left or right invariant then this measure class is unique and is both left and right invariant. Moreover the Borel group is standard.*

THEOREM 7.2. *Let G be a subgroup of the separable locally compact group $\mathcal{G}$. Give the space $\mathcal{G}/G$ of right G cosets Gx the quotient Borel structure. Let there exist a Borel set N in $\mathcal{G}/G$ such that $(\mathcal{G}/G)-N$ is countably separated and such that the inverse image of N in $\mathcal{G}$ has Haar measure zero. Then G is closed. Conversely if G is closed then $\mathcal{G}/G$ is standard.*

Proof [4]. If G is closed then by Lemma 1.1 of [11] there exists a Borel set E in $\mathcal{G}$ such that the canonical map h of $\mathcal{G}$ on $\mathcal{G}/G$ is one-to-one on E and such that $h(E)=\mathcal{G}/G$. Hence, by Theorem 3.2, $\mathcal{G}/G$ is standard.

Now omit the hypothesis that G is closed and let N be the subset of $\mathcal{G}/G$ described in the statement of the theorem. We show first that we may restrict attention to the case in which $\overline{G}=\mathcal{G}$. To do this let μ be a finite member of the unique invariant measure class in $\mathcal{G}$ and let h_1 be the canonical map of $\mathcal{G}$ on $\mathcal{G}/\overline{G}$. Since $\mathcal{G}/\overline{G}$ is standard and hence countably separated we may decompose μ as a direct integral of measures μ_x on the right cosets $\overline{G}x$ as described in Lemma 11.1 of [11]. Applying this lemma further we conclude that for almost all right cosets $\overline{G}x$ (with respect to the quotient measure $\tilde{\mu}$ in $\mathcal{G}/\overline{G}$), $\mu_x(h^{-1}(N)\cap\overline{G}x)=0$. Applying Lemma 11.5 of [11] we conclude that for $\tilde{\mu}$ almost all right cosets $\overline{G}x$ the measure class C_{μ_x} is invariant under left multiplication by members of $\overline{G}$. Hence there exists x_0 such that $\mu_{x_0}(h^{-1}(N)\cap\overline{G}x_0)=0$ and such that $C_{\mu_{x_0}}$ is invariant under left multiplication by members of $\overline{G}$. The mapping $y\rightarrow yx_0^{-1}$ is a Borel isomorphism of $\overline{G}x_0$ onto $\overline{G}$ which carries $C_{\mu_{x_0}}$ into a left invariant measure class in $\overline{\mathcal{G}}$ and $h^{-1}(N)\cap\overline{G}x_0$ into $h^{-1}(N)x_0\cap\overline{G}$. It follows that $h^{-1}(N)x_0\cap\overline{G}$ is a Borel subset of $\overline{G}$ with $\overline{G}$ Haar measure zero and is of the form $h_2^{-1}(N')$ where N' is a Borel subset of $\overline{G}/G$ and h_2 is the canonical mapping of $\overline{G}$ on $\overline{G}/G$. Moreover $\overline{G}-h_2^{-1}(N')$ is the image under this same Borel isomorphism of $\overline{G}x_0-(h^{-1}(N)\cap\overline{G}x_0)$. Thus $(\overline{G}/G\cdot)-N'$ is the image under a Borel isomorphism of a subspace of $(\mathcal{G}/G)-N$ and hence is countably separated. Hence if we replace $\mathcal{G}$ by $\overline{G}$ the hypotheses of our theorem are still satisfied. It follows at once that we need only consider the case in which $\overline{G}=\mathcal{G}$.

Suppose then that $\overline{G}=\mathcal{G}$ and let μ be a finite measure in the unique invariant measure class in $\mathcal{G}$. We shall derive a contradiction from the assumption that $G\neq\mathcal{G}$. We show first that for every set $h^{-1}(E)$ where E is a Borel set in $\mathcal{G}/G$ and h is as above, either $h^{-1}(E)$ or $\mathcal{G}-h^{-1}(E)$ has measure zero. Indeed let ϕ be the characteristic function of $h^{-1}(E)$. Then $\phi(tx)=\phi(x)$ for all $t\in G$ and all $x\in\mathcal{G}$. Hence for all $f\in\mathcal{L}^1(\mathcal{G})$, $\int f(x)\phi(tx)dx=\int f(x)\phi(x)dx$ for all $t\in G$ where dx denotes left invariant Haar measure in $\mathcal{G}$. Hence $\int f(t^{-1}x)\phi(x)dx=\int f(x)\phi(x)dx$ for all $t\in G$ and all $f\in\mathcal{L}^1(\mathcal{G})$. Since G is dense in $\mathcal{G}$ and left translation is continuous in the $\mathcal{L}^1$ norm we conclude that $\int f(x)\phi(yx)dx=\int f(x)\phi(x)dx$ for all $y\in\mathcal{G}$ and all $f\in\mathcal{L}^1(\mathcal{G})$. Hence for all $y\in\mathcal{G}$, $\phi(yx)=\phi(x)$

(4) I am indebted to P. R. Halmos for a suggestion which led to a simplification of this proof.

for almost all $x \in \mathcal{G}$. Hence for almost all $x \in \mathcal{G}$ $\phi(yx) = \phi(x)$ for almost all $y \in \mathcal{G}$. Hence ϕ is almost everywhere constant. Hence either $h^{-1}(E)$ or $\mathcal{G} - h^{-1}(E)$ is a null set as was to be proved.

Finally let $E_1, E_2, \cdots$ be a countable separating family for $(\mathcal{G}/G) - N$ and let $F_j = h^{-1}(E_j)$. Each right coset Gx not in $h^{-1}(N)$ is the intersection of the F_j which contain it. Moreover, by what we have just proved, there exists at most one coset Gx such that $\mu(Gx) = 0$. Since the null sets of μ are translation invariant and $G \neq \mathcal{G}$ it follows that $\mu(Gx) = 0$ for every $x \in \mathcal{G}$. Since $\mu(F_j) = 0$ or $\mu(\mathcal{G})$ for all j it follows that every Gx is either in $h^{-1}(N)$ or is in some F_j with $\mu(F_j) = 0$. Hence $\mathcal{G}$ is the union of $h^{-1}(N)$ and the F_j of measure zero. Hence $\mu(\mathcal{G}) = 0$. This contradiction completes the proof of the theorem.

8. **Borel structure in the duals of algebras.** Let $\mathfrak{A}$ be a separable Banach *-algebra with an identity; that is a separable Banach algebra over the complex numbers equipped with an involutary conjugate linear anti-automorphism $f \to f^*$. By a *representation* L of $\mathfrak{A}$ we shall mean a homomorphism $f \to L_f$ of $\mathfrak{A}$ into the algebra $\mathfrak{B}(\mathfrak{H}(L))$ of all bounded linear operators of some separable Hilbert space $\mathfrak{H}(L)$ into itself such that (a) $(L_f)^* = L_{f^*}$ for all $f \in \mathfrak{A}$, and (b) $\|L_f\| \leq \|f\|$ for all $f \in \mathfrak{A}$. Let L and M be representations of $\mathfrak{A}$. If there exists a unitary map U of $\mathfrak{H}(L)$ on $\mathfrak{H}(M)$ such that $UL_fU^{-1} = M_f$ for all $f \in \mathfrak{A}$ we say that L and M are *equivalent* and write $L \simeq M$. If no closed proper subspace $\mathfrak{H}'$ of $\mathfrak{H}(L)$ is such that $L_f(\mathfrak{H}') \subseteq \mathfrak{H}'$ for all $f \in \mathfrak{A}$ we say that L is *irreducible*. We denote the set of all equivalence classes of representations of $\mathfrak{A}$ by $\mathfrak{A}^r$ and the set of all equivalence classes of irreducible representations of $\mathfrak{A}$ by $\widehat{\mathfrak{A}}$. We call $\widehat{\mathfrak{A}}$ the *dual* of $\mathfrak{A}$. It is the purpose of this section to introduce a natural Borel structure into $\widehat{\mathfrak{A}}$ and study some of its properties.

For each $n = 1, 2, 3, \cdots$ let $\mathfrak{H}_n$ denote the n dimensional Hilbert space of n-tuples of complex numbers where $\|(c_1, c_2, \cdots, c_n)\|^2 = \sum_{k=1}^{n} |c_k|^2$. Let $\mathfrak{H}_\infty$ denote the classical Hilbert space of all infinite sequences $c_1, c_2, \cdots$ of complex numbers such that $\sum_{k=1}^{\infty} |c_k|^2 < \infty$ and $\|(c_1, c_2, \cdots)\|^2 = \sum_{k=1}^{\infty} |c_k|^2$. Let $\mathfrak{A}^c$ (c stands for concrete as opposed to abstract) denote the set of all representations L of $\mathfrak{A}$ such that $\mathfrak{H}(L)$ is one of the spaces $\mathfrak{H}_1, \mathfrak{H}_2, \cdots, \mathfrak{H}_\infty$ and where *we do not identify equivalent representations.* For each $L \in \mathfrak{A}^c$ let L^Δ denote the equivalence class to which it belongs. Then $L \to L^\Delta$ is a map of $\mathfrak{A}^c$ *onto* $\mathfrak{A}^r$. We give $\mathfrak{A}^r$, $\widehat{\mathfrak{A}}$ and $\mathfrak{A}^c$ Borel structures as follows. We denote by $\mathfrak{A}_n^c$ the subset of $\mathfrak{A}^c$ consisting of all L with $\mathfrak{H}(L) = \mathfrak{H}_n$ and give $\mathfrak{A}^c$ the smallest Borel structure having the following two properties: (a) $\mathfrak{A}_n^c$ is a Borel subset of $\mathfrak{A}^c$ for all n. (b) For each $n = 1, 2, \cdots, \infty$ each ϕ and ψ in $\mathfrak{H}_n$ and each $f \in \mathfrak{A}$, $L \to (L_f(\phi), \psi)$ is a Borel function on $\mathfrak{A}_n^c$. We give $\mathfrak{A}^r$ the smallest Borel structure such that $L \to L^\Delta$ is a Borel function; i.e. we make $\mathfrak{A}^r$ a quotient Borel space of $\mathfrak{A}^c$ with respect to the equivalence relation defined above. Finally we give $\widehat{\mathfrak{A}}$ the Borel structure which it inherits as a subset of $\mathfrak{A}^r$.

THEOREM 8.1. *$\mathfrak{A}^c$ is a standard Borel space.*

Proof. It will suffice to prove that each $\mathfrak{A}_n^c$ is a standard Borel space. Let $\mathfrak{F}_a^n$ denote the set of all linear transformations from $\mathfrak{A}$ to $\mathfrak{B}(\mathfrak{H}_n)$ which satisfy (b) of the definition of a representation. Give $\mathfrak{F}_a^n$ the smallest Borel structure such that all functions $L\to(L_f(\phi), \psi)$ are Borel functions. Then $\mathfrak{A}_n^c$ is a subset of $\mathfrak{F}_a^n$ and its Borel structure is that it inherits as a subspace of $\mathfrak{F}_a^n$. Thus it will suffice to show that $\mathfrak{F}_a^n$ is a standard Borel space and that $\mathfrak{A}_n^c$ is a Borel subset of $\mathfrak{F}_a^n$. Let $f_1, f_2, \cdots$ be a countable subset of linearly independent elements of $\mathfrak{A}$ such that $f_1\dotplus f_2\dotplus \cdots$ is dense in $\mathfrak{A}$. Then $L\in\mathfrak{F}_a^n$ is uniquely determined as soon as $L_{f_1}, L_{f_2}, \cdots$ are known. Moreover given a sequence $A_1, A_2, \cdots$ of members of $\mathfrak{B}(\mathfrak{H}_n)$ there exists $L\in\mathfrak{F}_a^n$ such that $L_{f_j}=A_j$ for all j if and only if

$$(*)\qquad \|r_1A_1 + r_2A_2 + \cdots + r_nA_n\| \leq \|r_1f_1 + r_2f_2 + \cdots + r_nf_n\|$$

for all finite linear combinations with complex rational coefficients r_j of the f_j. We give the space of all sequences $A_1, A_2, \cdots$ of members of $\mathfrak{B}(\mathfrak{H}_n)$ the smallest Borel structure for which the function $A_1, A_2, \cdots \to(A_j(\phi), \psi)$ is a Borel function for all $j=1, 2, \cdots$ and all ϕ and $\psi\in\mathfrak{H}_n$. We call this space S_n. S_n is then the Cartesian product of countably many replicas of the space $\mathfrak{B}(\mathfrak{H}_n)$ where $\mathfrak{B}(\mathfrak{H}_n)$ has the smallest Borel structure for which all functions $A\to(A(\phi), \psi)$ are Borel functions; that is the Borel structure defined by the weak topology of $\mathfrak{B}(\mathfrak{H}_n)$. But $\mathfrak{B}(\mathfrak{H}_n)$ is a union of the countably many sets $Z_{n,m}$ where $Z_{n,m}$ is the set of all $A\in\mathfrak{B}(\mathfrak{H}_n)$ with $\|A\|\leq m$. Each of these sets is a compact metric space in the weak topology and hence is a standard Borel space. Thus $\mathfrak{B}(\mathfrak{H}_n)$ and hence S_n are standard Borel spaces. Now $L\to L_{f_1}, L_{f_2}, \cdots$ is evidently a one-to-one mapping of $\mathfrak{F}_a^n$ onto the subset of all members $A_1, A_2, \cdots$ of S_n which satisfy the inequalities (*). Moreover there are only countably many such inequalities and for fixed $r_1, r_2, \cdots, r_n$, $\|r_1A_1+r_2A_2+ \cdots +r_nA_n\|$ is clearly a Borel function of the sequence $A_1, A_2, \cdots$. Hence the subset of S_n consisting of all sequences $A_1, A_2, \cdots$ which satisfy the inequalities (*) is a Borel subset of a standard Borel space and hence is a standard Borel space. Thus to complete the proof that $\mathfrak{F}_a^n$ is standard we need only show that $L\to L_{f_1}, L_{f_2}, \cdots$ is a Borel isomorphism. This amounts to proving that the smallest Borel structure for which the functions $L\to(L_f(\phi), \psi)$ are all Borel functions is identical with that for which the functions $L\to(L_{f_j}(\phi), \psi)$ are all Borel functions. But $(L_f(\phi), \psi)$ is a limit of functions of the form $c_1(L_{f_1}(\phi), \psi)+c_2(L_{f_2}(\phi), \psi)+ \cdots +c_n(L_{f_n}(\phi), \psi)$ and every limit of Borel functions is a Borel function. Thus $L\to L_{f_1}, L_{f_2}, \cdots$ is a Borel isomorphism and the proof that $\mathfrak{F}_a^n$ is standard is complete. It remains to prove that $\mathfrak{A}_n^c$ is a Borel subset of $\mathfrak{F}_a^n$. Let $\phi_1, \phi_2, \cdots$ be an orthonormal basis for $\mathfrak{H}_n$. Then $L\in\mathfrak{F}_a^n$ is a ring homomorphism if and only if $(L_{f_if_j}(\phi_r), \phi_s)$ $=(L_{f_i}L_{f_j}(\phi_r), \phi_s)$ for all i, j, r, and s; that is if and only if $(L_{f_if_j}(\phi_r), \phi_s)$ $=(L_{f_j}(\phi_r), L_{f_i}^*(\phi_s))$ for all i, j, r, and s. But the left hand member of this equality is a Borel function of L and the right hand member being equal to

$$\sum_{k=1}^{\infty} (L_{f_j}(\phi_r), \phi_k)\left[(L_{f_i}(\phi_k), \phi_s)\right]^{\text{conj.}}$$

is also a Borel function of L. Thus the set of all members of $\mathfrak{F}_a^n$ which are ring homomorphisms is a Borel subset of $\mathfrak{F}_a^n$. Quite analogous arguments show that the set of all $L \in \mathfrak{F}_a^n$ such that $L_{f*} = (L_f)^*$ for all $f \in \mathfrak{A}$ is a Borel subset of $\mathfrak{F}_a^n$. Thus the set of all representations in $\mathfrak{F}_a^n$ is a Borel subset of $\mathfrak{F}_a^n$. Thus $\mathfrak{A}^c$ is a standard Borel space as was to be proved.

An *intertwining operator* T for a pair of representations L and M is a bounded linear transformation from $\mathfrak{H}(L)$ to $\mathfrak{H}(M)$ such that $TL_f = M_f T$ for all $f \in \mathfrak{A}$. We denote the vector space of all intertwining operators for L and M by $\mathfrak{R}(L, M)$. Its dimension we call the *interwining number* of L and M and denote by $\mathfrak{g}(L, M)$. As is well known and easily proved $\mathfrak{g}(L, L) = 1$ if and only if L is irreducible.

THEOREM 8.2. *Let S be a Borel space and let f and g be Borel functions from S to $\mathfrak{A}^c$. Then $\mathfrak{g}(f(y), g(y))$ is a Borel function of y.*

Proof. The proof is an obvious adaptation of the proof of Theorem 2.8 of [12]. We leave details to the reader.

THEOREM 8.3. *The set of all irreducible members of $\mathfrak{A}^c$ is a Borel subset of $\mathfrak{A}^c$ and hence as a Borel space is standard.*

Proof. By Theorem 8.2, $\mathfrak{g}(L, L)$ is a Borel function of L. Hence the set where $\mathfrak{g}(L, L) = 1$ is a Borel set.

THEOREM 8.4. *$\widehat{\mathfrak{A}}$ is a separated Borel space. Indeed every point is a Borel set. If $\widehat{\mathfrak{A}}$ is countably separated it is analytic.*

Proof. To prove the first two statements we need only show that the set of all $L \in \mathfrak{A}^c$ equivalent to a fixed irreducible member L_0 of $\mathfrak{A}^c$ is a Borel set. But L and L_0 are equivalent if and only if L is irreducible and $\mathfrak{g}(L, L_0) = 1$. We now need only apply Theorems 8.2 and 8.3. To prove the second statement we observe that $\widehat{\mathfrak{A}}$ is by Theorem 8.3 a Borel image of a standard Borel space. Hence, by Theorem 5.1, if $\widehat{\mathfrak{A}}$ is countably separated it is analytic.

If $\widehat{\mathfrak{A}}$ is countably separated and hence analytic we shall say that it is *smooth* or that $\mathfrak{A}$ has a *smooth dual*. Smoothness for $\widehat{\mathfrak{A}}$ is clearly equivalent to the existence of countably many Borel functions defined on the irreducible members of $\mathfrak{A}$ constant on the equivalence classes and serving to distinguish between any two equivalence classes; that is to the possibility of describing the equivalence classes of irreducible representations by means of countably many Borel invariants. Theorem 5.2 is also applicable in the present context and gives the following sufficient condition for smoothness.

THEOREM 8.5. *Let there exist in $\mathfrak{A}^c$ a Borel set S which has just one point in common with each equivalence class of irreducible members of $\mathfrak{A}^c$. Then $\widehat{\mathfrak{A}}$ is smooth.*

Proof. Let $\mathcal{G}$ denote the group of all unitary operators in $\mathcal{B}(\mathcal{H}_\infty)$. Give $\mathcal{G}$ the Borel structure it inherits from the Borel structure in $\mathcal{B}(\mathcal{H}_\infty)$ described in the proof of Theorem 8.1. Choosing the natural basis in $\mathcal{H}_\infty$ and describing the condition that $U \in \mathcal{B}(\mathcal{H}_\infty)$ be unitary in terms of its matrix with respect to this basis we verify at once that $\mathcal{G}$ is a Borel subset of $\mathcal{B}(\mathcal{H}_\infty)$ and hence is standard as a Borel space. Moreover in the product $U_1 U_2^{-1}$ the i, jth matrix element is an infinite sum of products of Borel functions on $\mathcal{G} \times \mathcal{G}$ and hence is itself a Borel function. Now by Theorem 3.2 the mapping m taking $U \in \mathcal{G}$ into its matrix is a Borel isomorphism of $\mathcal{G}$ with a subset of the Cartesian product of countably many replicas of the unit disk in the complex plane. Since the i, jth matrix element is a Borel function for all i and j, $U_1, U_2 \to m(U_1 U_2^{-1})$ is a Borel function on $\mathcal{G} \times \mathcal{G}$ hence U_1, U_2^{-1} is a Borel function from $\mathcal{G} \times \mathcal{G}$ to $\mathcal{G}$. Thus $\mathcal{G}$ is a Borel group. Similarly one proves that $U, L \to ULU^{-1}$, where $(ULU^{-1})_f = UL_f U^{-1}$ for all $f \in \mathcal{A}$, is a Borel function from $\mathcal{G} \times \mathcal{A}^c_\infty$ to $\mathcal{A}^c_\infty$. Thus the hypotheses of Theorem 5.2 are satisfied and we deduce that the set $\widehat{\mathcal{A}}_\infty$ of all infinite dimensional members of $\widehat{\mathcal{A}}$ is an analytic Borel space. A similar argument applies to each $\widehat{\mathcal{A}}_n$ (or alternatively we may apply Theorem 8.7 below). Since the union of disjoint analytic Borel spaces is clearly analytic the proof is complete.

Like its predecessor this theorem relates smoothness to the possibility of describing the equivalence classes of irreducible representations in a suitably regular manner. It says that $\widehat{\mathcal{A}}$ is smooth whenever there exists a Borel set of "canonical forms" for concrete irreducible representations under the relation of equivalence. Here however we are unable to prove a strict converse. The existence of canonical forms is in fact more closely related to a slight weakening of smoothness. Note that a countably separated finite measure in $\widehat{\mathcal{A}}$ is necessarily standard. Indeed if $\mu(N) = 0$ where $\widehat{\mathcal{A}} - N$ is countably separated and N is a Borel set then the inverse image of $\widehat{\mathcal{A}} - N$ in $\mathcal{A}^c$ is a Borel set. Hence by Theorem 8.3 it is standard. Hence by Theorem 5.1 $\widehat{\mathcal{A}} - N$ is analytic. Hence by Theorem 6.1 μ is standard. If every finite (or σ-finite) Borel measure in $\widehat{\mathcal{A}}$ is countably separated and hence standard we shall say that $\widehat{\mathcal{A}}$ is *metrically smooth* or that $\mathcal{A}$ has a *metrically smooth dual.*

THEOREM 8.6. *$\widehat{\mathcal{A}}$ is metrically smooth if and only if for each finite Borel measure μ in $\widehat{\mathcal{A}}$ there exists a Borel subset N of $\widehat{\mathcal{A}}$ and a Borel subset S of $\mathcal{A}^c$ such that $\mu(N) = 0$ and such that S contains one and only one member of each equivalence class in $\widehat{\mathcal{A}} - N$.*

Proof. Let μ be a finite Borel measure in $\widehat{\mathcal{A}}$ and suppose that N and S exist as in the statement of the theorem. Let $\mathcal{A}^{c'}$ denote the set of all $L \in \mathcal{A}^c$ whose equivalence class is in $\widehat{\mathcal{A}} - N$. $\mathcal{A}^{c'}$ is then a Borel subset of $\mathcal{A}^c$. Since $\mathcal{A}^c$ is standard, $\mathcal{A}^{c'}$ as a Borel space is standard. The argument of Theorem 8.5 applies to $\mathcal{A}^{c'}$ and tells us that $\widehat{\mathcal{A}} - N$ is analytic. Hence μ is countably separated. Hence the condition of the theorem is sufficient for the metric smoothness of $\hat{\mathfrak{A}}$. Suppose conversely that $\hat{\mathfrak{A}}$ is metrically smooth. Let μ be a finite Borel measure in $\hat{\mathfrak{A}}$ and let N_1 be a Borel set in $\hat{\mathfrak{A}}$ such that $\mu(N_1) = 0$ and $S_1 = \hat{\mathfrak{A}} - N_1$ is a standard subspace. Let S_2 be the set of all $L \in \mathfrak{A}^c$ whose equivalence class is in S_1. Then S_2 as a Borel space is standard. Let A be the set of all pairs L^Δ, L where $L \in S_2$ and L^Δ is the equivalence class of L. Since $L \to L^\Delta$ is a Borel function and S_1 and S_2 are standard it follows from [6, p. 291] that A is a Borel subset of $S_1 \times S_2$. Applying Theorem 6.3 we conclude the existence of a Borel subset N_2 of S_1 and a Borel function ϕ from $S_1 - N_2$ to S_2 such that $\mu(N_2) = 0$ and $\phi(L)^\Delta = L$ for all $L \in S_1 - N_2$. Since ϕ is one-to-one its image S in S_2 is a Borel set by Theorem 3.2. Clearly S contains one and only one member of each equivalence class in $\hat{\mathfrak{A}} - (N_1 \cup N_2)$. Taking $N = N_1 \cup N_2$ the proof of the theorem is complete.

The interest of Theorem 8.6 is heightened by the fact that it is metric smoothness rather than smoothness that plays a principal role in the decomposition theory to be described in §10.

When only finite dimensional irreducible representations are involved one can use the notion of the "character" of a representation to obtain Borel invariants and we have.

THEOREM 8.7. *For each finite integer n, $\hat{\mathfrak{A}}_n$ the set of all n dimensional members of $\hat{\mathfrak{A}}$ is an analytic Borel space.*

Proof. For each $f \in \mathfrak{A}$ define a function $\hat{f}$ on $\bigcup_{n=1}^{\infty} \hat{\mathfrak{A}}_n$ by setting $\hat{f}(L^\Delta) = \text{Trace}\ (L_f)$. If $\phi_1, \phi_2, \cdots, \phi_n$ is a basis for $\mathfrak{H}_n$ then $\hat{f}(L^\Delta) = (L_f(\phi_1), \phi_1) + \cdots + (L_f(\phi_n), \phi_n)$. Since the right hand member of the equation is a Borel function of L, $\hat{f}$ is a Borel function on $\hat{\mathfrak{A}}_n$ for each $n = 1, 2, \cdots$. Thus to show that $\hat{\mathfrak{A}}_n$ is countably separated it will suffice to show that there exist countably many elements $f_1, f_2, \cdots$ in $\mathfrak{A}$ such that $L^\Delta \neq M^\Delta$ implies that $\hat{f}_j(L^\Delta) \neq \hat{f}_j(M^\Delta)$ for some j. Let $f_1, f_2, \cdots$ be any countable dense subset of $\mathfrak{A}$. Suppose that $\hat{f}_j(L^\Delta) = \hat{f}_j(M^\Delta)$ for all j. Then Trace L_{f_j} = Trace M_{f_j} for all j. But Trace L_f and Trace M_f are continuous functions of f. Hence Trace L_f = Trace M_f for all f. Let I_L be the ideal in $\mathfrak{A}$ on which $L_f = 0$. Since L is irreducible $\mathfrak{A}/I_L$ is a full matrix algebra. Hence $L_f = 0$ if and only if Trace $L_f L_g = 0$ for all g; that is if and only if Trace $L_{fg} = 0$ for all g. Similarly $M_f = 0$ if and only if Trace $M_{fg} = 0$ for all g. Hence $I_L = I_M$. Hence $L^\Delta = M^\Delta$ and the proof of the theorem is complete

COROLLARY. *If $\mathfrak{A}$ has only finite dimensional irreducible representations then $\hat{\mathfrak{A}}$ is smooth.*

The question arises as to the extent to which the argument of Theorem 8.7 may be extended so as to give useful sufficient conditions for the smoothness of $\hat{\mathfrak{A}}$ even when infinite dimensional irreducible representations of $\mathfrak{A}$ exist. The trace is defined for the members of a reasonably large class of operators in infinite dimensional spaces and one may hope to prove generalizations of the preceding corollary in which the hypothesis that all irreducible representations are finite dimensional is replaced by one ensuring only that sufficiently many traces exist. For example one might prove in this way that the CCR algebras of Kaplansky [5] have smooth duals or that the same is true for every algebra with sufficiently many type I representations which are normal in the sense of Godement [2]. We hope to return to these questions at another time.

9. **Borel structure in the duals of groups.** Let $\mathcal{G}$ be a separable locally compact group. By a *representation* L of $\mathcal{G}$ we shall mean a homomorphism $x \to L_x$ of $\mathcal{G}$ into the group $\mathcal{V}(\mathcal{H}(L))$ of all unitary mappings of some separable Hilbert space $\mathcal{H}(L)$ onto itself such that for all ϕ and $\psi \in \mathcal{H}(L)$ the mapping $x \to (L_x(\phi), \psi)$ is a Borel function on $\mathcal{G}$. As is well known and easily verified this implies that $x \to L_x(\phi)$ is a continuous function from $\mathcal{G}$ to $\mathcal{H}(L)$ for all $\phi \in \mathcal{H}(L)$. Such representations are usually called strongly continuous unitary representations but as we shall deal with no other kind we shall refer to them simply as representations. Just as in the case of algebras the representations L and M are said to be *equivalent* and we write $L \simeq M$ if there exists a unitary map U of $\mathcal{H}(L)$ on $\mathcal{H}(M)$ such that $UL_xU^{-1} = M_x$ for all $x \in \mathcal{G}$. Moreover if no closed proper subspace $\mathcal{H}'$ of $\mathcal{H}(L)$ is such that $L_x(\mathcal{H}') \subseteq \mathcal{H}'$ for all $x \in \mathcal{G}$ we say that L is *irreducible*. We denote the set of all equivalence classes of representations of $\mathcal{G}$ by $\mathcal{G}^r$ and the set of all equivalence classes of irreducible representations of $\mathcal{G}$ by $\widehat{\mathcal{G}}$. We call $\widehat{\mathcal{G}}$ the dual of $\mathcal{G}$. We define the concrete Hilbert spaces $\mathcal{H}_1, \mathcal{H}_2, \cdots, \mathcal{H}_\infty$ exactly as in §8 and define Borel spaces $\mathcal{G}_n^c$ and $\mathcal{G}^c$ by strict analogy with the definitions of $\mathfrak{A}_n^c$ and $\mathfrak{A}^c$. Continuing the parallel we define L^Δ as the equivalence class to which L belongs and define Borel structures in $\mathcal{G}^r$ and $\widehat{\mathcal{G}}$ by regarding $\mathcal{G}^r$ as a quotient space of $\mathcal{G}^c$ and $\widehat{\mathcal{G}}$ as a subspace of $\mathcal{G}^r$.

Consider now the separable Banach *-algebra $\mathfrak{A}_\mathcal{G}$ consisting of all equivalence classes of complex valued functions on $\mathcal{G}$ which are summable with respect to right invariant Haar measure in $\mathcal{G}$ where multiplication is defined by $(f * g)(x) = \int f(xy^{-1})g(y)dy$ and the * operator by $f^*(x) \to f(x^{-1})/\Delta(x)$, Δ being the real valued function on $\mathcal{G}$ such that $\int g(y)dy = \int (g(y^{-1})/\Delta(y))dy$ for all summable g. For the proof that $\mathfrak{A}_\mathcal{G}$ is indeed a *-algebra see Loomis [7, p. 120]. As is well known (Loomis [7, #32]) if L is any representation of $\mathcal{G}$ then there is a unique representation L^0 of $\mathfrak{A}_\mathcal{G}$ such that $\mathcal{H}(L^0) = \mathcal{H}(L)$ and $(L_f^0(\phi), \psi) = \int f(y)(L_y(\phi), \psi)$ for all $f \in \mathfrak{A}_\mathcal{G}$ and all ϕ and $\psi \in \mathcal{H}(L)$. Moreover this map is one-to-one and onto from the set of all representations of $\mathcal{G}$ to the set of all representations M of $\mathfrak{A}_\mathcal{G}$ which are *proper* in the sense that the linear union of the ranges of the M_f is dense in $\mathcal{H}(M)$. It preserves irreducibility, dimension, and equivalence. Let us denote by $\mathfrak{A}^{cp}$, $\widehat{\mathfrak{A}}^p \mathfrak{A}^{rp}$ etc. the subsets of $\mathfrak{A}^c$, $\widehat{\mathfrak{A}}$, $\mathfrak{A}^r$ etc. consisting of the proper representations or equivalence classes of representations which they contain. It is clear that the representation M is proper if and only if $\mathcal{J}(M, Z) = 0$ where Z is the representation such that $\mathcal{H}(Z)$ is one dimensional and $Z_f=0$ for all f. Thus by Theorem 8.2, $\mathfrak{A}^{cp}$, $\widehat{\mathfrak{A}}^p\mathfrak{A}^{rp}$ etc. are Borel subsets of $\mathfrak{A}^c$, $\widehat{\mathfrak{A}}\mathfrak{A}^r$, etc.

THEOREM 9.1. *The mapping $L\rightarrow L^0$ of $\mathcal{G}^c$ on $\mathfrak{A}_{\mathcal{G}}^{cp}$ is a Borel isomorphism.*

Proof. Let $\mathcal{G}_n^c$ be the set of all members L of $\mathcal{G}^c$ with $\mathcal{H}(L)=\mathcal{H}_n$. It will clearly suffice to show that $L\rightarrow L^0$ is a Borel isomorphism of $\mathcal{G}_n^c$ on $(\mathfrak{A}_{\mathcal{G}})_n^{cp}$ for each $n=1, 2, \cdots$. Choose a fixed n and let $\mathfrak{F}_1^n$ be the set of all Borel functions on $\mathcal{G}_n^c$ of the form $L\rightarrow(L_x(\phi), \psi)$ where $x\in\mathcal{G}$ and ϕ and ψ are in $\mathcal{H}_n$. Let $\mathfrak{F}_{-1}^n$ be the set of all Borel functions on $\mathcal{G}_n^c$ of the form $L\rightarrow(L_f^0(\phi), \psi)$ where $f\in\mathfrak{A}_{\mathcal{G}}$ and ϕ and ψ are in $\mathcal{H}_n$. We need only show that for $j=1$ or -1 every member of $\mathfrak{F}_{-j}^n$ is a Borel function with respect to the smallest Borel structure $\mathcal{B}_j$ in $\mathcal{G}_n^c$ for which all members of $\mathfrak{F}_j^n$ are Borel functions. Choose a sequence $f_1, f_2, \cdots$ of members of $\mathfrak{A}_{\mathcal{G}}$ such that $f_n\geq 0$ for all n $\int f_n(y)dy=1$ and f_n vanishes outside of a set containing the identity e and of diameter less than $1/n$. (Since $\mathcal{G}$ is separable we may suppose it equipped with a metric.) Then $(L_{f_n}^0(\phi), \psi) =\int f_n(y)(L_y(\phi), \psi)dy$ for all ϕ and ψ in $\mathcal{H}_n$ and since $(L_y(\phi), \psi)$ is a continuous function of y the right hand member has (ϕ, ψ) as a limit when n tends to infinity. Now let $f_n^x(y) =f_n(yx^{-1})$. Then

$$(L_{f_n^x}(\phi), \psi) = \int f_n(yx^{-1})(L_y(\phi), \psi)dy = \int f_n(y)(L_yL_x(\phi), \psi)dy$$
$$= (L_{f_n}(L_x(\phi)), \psi).$$

Hence $\lim_{n\rightarrow\infty} (L_{f_n^x}(\phi), \psi) = L_x(\phi), \psi)$. Hence for any Borel structure in which all members of $\mathfrak{F}_{-1}^n$ are Borel functions so are all members of $\mathfrak{F}_1^n$. On the other hand for each $f\in\mathfrak{A}_{\mathcal{G}}$ we may choose f' so that f' is a Borel function and $f(y) =f'(y)$ for almost all y. Hence

$$(L_f^0(\phi), \psi) = \int f'(y)(L_y(\phi), \psi)dy.$$

Let $\mathcal{G}_n^c$ be given the Borel structure $\mathcal{B}_1$. Then since $(L_y(\phi), \psi)$ is continuous in y for each fixed L, ϕ and ψ it follows from the argument of Lemma 9.2 of [11] that $(L_y(\phi), \psi)$ is a Borel function on $\mathcal{G}\times\mathcal{G}_n^c$ for each ϕ and ψ in $\mathcal{H}_n$. Hence by the Fubini theorem $\int f'(y)(L_y(\phi), \psi)$ is a Borel function on $\mathcal{G}_n^c$ for each ϕ and ψ in $\mathcal{H}_n$. Thus each member of $\mathfrak{F}_{-1}^n$ is a Borel function with respect to $\mathcal{B}_1$. This completes the proof.

COROLLARY. *Theorems* 8.1 *through* 8.6 *remain true when $\mathfrak{A}^c$ and $\widehat{\mathfrak{A}}$ are replaced by $\mathcal{G}^c$ and $\widehat{\mathcal{G}}$ respectively. Moreover the mapping of $\widehat{\mathcal{G}}$ on $\widehat{\mathfrak{A}}_{\mathcal{G}}^p$ induced by $L\rightarrow L^0$ is a Borel isomorphism.*

10. **Direct integrals of representations.** In this section $\mathcal{W}$ will denote a fixed separable locally compact group or a fixed separable Banach *-algebra. The arguments will be of a quite general nature and it will not be necessary

to specify whether $\mathcal{W}$ is a group or an algebra. Let μ be a finite standard Borel measure in the Borel space S, and let $y \rightarrow L^y$ denote a Borel function from S to $\mathcal{W}^c$. We define what we shall call the direct integral $M = \int_S L^y d\mu(y)$ of the L^y with respect to μ as follows. For each $n = \infty, 1, 2, \cdots$ let S_n be the set of all $y \in S$ such that $\mathcal{H}(L^y) = \mathcal{H}_n$ and form the Hilbert space $\mathcal{H}(M)$ of all functions f from S to $\bigcup_{n=\infty,1,2\ldots} \mathcal{H}_n$ such that: (a) $f(y) \in \mathcal{H}_n$ for all $y \in S_n$. (b) For each $n = \infty, 1, 2, \cdots$, $(f(y), \phi)$ is a Borel function on S_n for all $\phi \in \mathcal{H}_n$. (c) $\int_S (f(y), f(y)) d\mu(y) < \infty$. We set

$$\|f\|^2 = \int_S (f(y), f(y)) d\mu(y)$$

and identify f and g whenever $\|f - g\| = 0$. For each $x \in \mathcal{W}$ and each $f \in \mathcal{H}(M)$ let $M_x(f) = g$ where $g(y) = L_x^y(f(y))$ for all $y \in S$. The function g is clearly in $\mathcal{H}(M)$ and $f \rightarrow g = M_x(f)$ is clearly a bounded linear operator in $\mathcal{H}(M)$. Finally there is no difficulty in verifying that $x \rightarrow M_x$ is a representation of $\mathcal{W}$ provided that one takes account of Lemma 9.2 of [11] in the case in which $\mathcal{W}$ is a group. It is obvious that $\int_{S'} L^y d\mu(y)$ and $\int_S L^y d\mu(y)$ are equivalent if S' is a Borel subset of S such that $\mu(S - S') = 0$ and it is easy to see that $\int_S L^y d\mu(y)$ and $\int_S L^y d\mu'(y)$ are equivalent whenever μ and μ' have the same null sets; i.e. whenever the measure classes C_μ and $C_{\mu'}$ are identical. In addition we have

THEOREM 10.1. *If ${}^1L^y$ and ${}^2L^y$ are equivalent for almost all y then $\int_S^1 L^y d\mu(y)$ and $\int_S^2 L^y d\mu(y)$ are equivalent.*

Proof. For each $y \in S$ there is a unique $n = n(y)$ such that $y \in S_n$. Let A_y denote the set of all unitary maps of $\mathcal{H}_{n(y)}$ on itself which set up an equivalence between ${}^1L^y$ and ${}^2L^y$. Suppose that there exists a Borel subset S' of S such that $\mu(S - S') = 0$ and a function $y \rightarrow V^y$ defined on S' such that $V^y \in A_y$ for all y and such that $(V^y(\phi), \psi)$ is a Borel function on $S' \cap S_{n(y)}$ for all n and all ϕ and $\psi \in \mathcal{H}_{n(y)}$. Then $f \rightarrow V(f)$, where $(V(f))(y) = V^y(f(y))$ for all y in S' and is zero otherwise, sets up an equivalence between $\int_S^1 L^y d\mu(y)$ and $\int_S^2 L^y d\mu(y)$. Thus our problem reduces to showing that V^y may be chosen from A_y so that the indicated measurability condition is satisfied. We do this by applying Theorem 6.3 with $S_1 = S_n$ and $S_2 = \mathcal{B}(\mathcal{H}_n)$ where the latter is given the Borel structure described in the proof of Theorem 8.1. Since we know that $\mathcal{B}(\mathcal{H}_n)$ is standard as a Borel space and that μ is a standard measure we have only to verify that the set of all pairs $y, T \in \mathcal{S}_n \times \mathcal{B}(\mathcal{H}_n)$ such that $T \in A_y$ is a Borel subset of $\mathcal{S}_n \times \mathcal{B}(\mathcal{H}_n)$. But this set is the intersection of the set of all y, T such that T is unitary with the set of all y, T such that $TL_x^y = L_x^y T$ for all $x \in \mathcal{W}$. Let us denote these sets by $\mathfrak{F}_1$ and $\mathfrak{F}_2$ respectively. It will of course suffice to prove that $\mathfrak{F}_1$ and $\mathfrak{F}_2$ are Borel sets. That $\mathfrak{F}_1$ is a Borel set follows at once from the first part of the proof of Theorem 8.5. To prove that $\mathfrak{F}_2$ is also a Borel set let $x_1, x_2, \cdots$ be a countable dense subset of $\mathcal{W}$ and let $\phi_1, \phi_2, \cdots$ be a fixed complete orthonormal set in $\mathcal{H}_n$. Then $y, T \in \mathfrak{F}_2$ if and only if

$(T({}^1L^y_{x_j}(\phi_i)), \phi_k) = ({}^2L^y_{x_j}(T(\phi_i)), \phi_k)$ for all i, j, and k. Moreover for each fixed i, j, and k we have

$$(T({}^1L^y_{x_j}(\phi_i)), \phi_k) = (L^y_{x_j}(\phi_i), T^*(\phi_k)) = \sum_r ({}^1L^y_{x_j}(\phi_i), \phi_r)(T(\phi_r), \phi_k)$$

which is clearly a Borel function on $S_n \times \mathcal{B}(\mathcal{H}_n)$. Since the right hand side may be shown to be a Borel function by the same argument it follows that $\mathfrak{F}_2$ is a Borel set.

It follows from the theorem just proved that $\int_S L^y d\mu(y)$ may be given a meaning when $y \rightarrow L^y$ is a Borel function from S to $\mathcal{W}^r$ instead of to $\mathcal{W}^c$. We simply choose any function $y \rightarrow {}^1L^y$ from S to $\mathcal{W}^c$ such that ${}^1L^y$ lies in the equivalence class L^y for almost all y and such that $y \rightarrow {}^1L^y$ is a Borel function and form $\int_S^1 L^y d\mu(y)$. The function $y \rightarrow {}^1L^y$ need not exist of course but if it does then $\int_S L^y d\mu(y)$ is a well defined member of $\mathcal{W}^r$. We say then that $y \rightarrow L^y$ is *integrable*. The next theorem gives a useful sufficient condition for integrability.

THEOREM 10.2. *Let $y \rightarrow L^y$ be a Borel function from S to $\mathcal{W}^r$ and let $\tilde{\mu}$ be the Borel measure in $\mathcal{W}^r$ obtained by setting $\tilde{\mu}(E) = \mu(E')$ where E' is the inverse image of E under the mapping $y \rightarrow L^y$. Then if $\tilde{\mu}$ is standard $y \rightarrow L^y$ is integrable.*

Proof. Let $\tilde{\mu}$ be standard and let E_0 be a Borel subset of $\mathcal{W}^r$ such that $\tilde{\mu}(\mathcal{W}^r - E_0) = 0$ and such that E_0 is standard as a subspace. Let g denote the Borel map of $\mathcal{W}^c$ on $\mathcal{W}^r$ which takes each L into its equivalence class L^Δ. Let $g^{-1}(E_0) = E_1$. Then g is a Borel function from E_1 to E_0 and E_0 and E_1 are both standard Borel spaces. Hence by [6, p. 291] the set of all pairs $L, g(L)$ is a Borel subset of $E_1 \times E_0$. We now apply Theorem 6.3 and deduce the existence of a $\tilde{\mu}$ null set N in E_0 and a Borel function h from $E_0 - N$ to E_1 such that $g(h(L)) = L$ for all $L \in E_0 - N$. We now set ${}^1L^y = h(L^y)$ for all $y \in E_0 - N$ and let ${}^1L^y$ be some fixed but arbitrary member of $\mathcal{W}^c$ for all y such that $L^y \in E_0 - N$. Then $y \rightarrow {}^1L^y$ is a Borel function and $g({}^1L^y) = L^y$ for almost all y. Thus $y \rightarrow L^y$ is integrable.

COROLLARY. *If $\mathcal{W}$ has a metrically smooth dual then every Borel function from S to $\widehat{\mathcal{W}}$ is integrable.*

It often happens that we are given a representation valued function $y \rightarrow L^y$ on a Borel space S equipped with a standard measure μ where the L^y are neither abstract (i.e. equivalence classes of representations) nor specifically in the concrete form which makes them members of $\mathcal{W}^c$. Instead each L^y is a concretely given representation in a concrete Hilbert space $\mathcal{H}(L^y)$ where the $\mathcal{H}(L^y)$ needn't be amongst the $\mathcal{H}_n$. The following theorem gives a useful necessary and sufficient condition for the integrability of the corresponding function $y \rightarrow L^y$ from S to $\mathcal{W}^r$ and hence for the existence of $\int_S L^y d\mu(y)$.

THEOREM 10.3. *Let $y \to L^y$ be a representation valued function on the Borel space S and let μ be a standard measure in S. Then there exists a Borel subset N of S such that $\mu(N)=0$ and such that the restriction to $S-N$ of $y \to (L^y)^\Delta$ is integrable if and only if there exists a sequence $f_1, f_2, \cdots$ of functions mapping S into $\bigcup_y \mathcal{H}(L^y)$ and having the following properties*: (a) *$f_j(y) \in \mathcal{H}(L^y)$ for all j and y.* (b) *For every function f which is a finite linear combination with constant coefficients of the f_j the function $y \to \|f(y)\|^2$ is μ summable.* (c) *For almost every $y \in S$ the sequence $f_1(y), f_2(y), \cdots$ of elements of $\mathcal{H}(L^y)$ has $\mathcal{H}(L^y)$ as its closed linear span.* (d) *For all j and k and all $x \in \mathcal{W}$ the function $y \to (L^y_x(f_j(y)), f_k(y))$ is μ measurable.*

Proof. Suppose that N exists and let $y \to {}^1L^y$ be a Borel function from $S-N$ to $\mathcal{W}^c$ such that L^y and ${}^1L^y$ are equivalent for all $y \in S-N'$ where N' is a Borel subset of S such that $N' \supseteq N$ and $\mu(N')=0$. Let U^y, for each $y \in S-N'$, be any unitary operator setting up the equivalence between ${}^1L^y$ and L^y. Let $g_1, g_2, \cdots$ be a sequence of functions defined as follows. If $y \in S-N'$ or if j is greater than the dimension of $\mathcal{H}({}^1L^y)$ let $g_j(y)=0$. Otherwise let $g_j(y)=e_j$ where e_j is the element of $\mathcal{H}({}^1L^y)$ which is one at the jth place and zero elsewhere (we recall that $\mathcal{H}({}^1L^y)$ is a space of sequences). Let $f_j(y) = U^y(g_j(y))$ for all $y \in S-N'$ and 0 for all $y \in N'$. It is immediate that the sequence $f_1, f_2, \cdots$ satisfies conditions (a), (b), (c) and (d) of the statement of the theorem. Now suppose conversely that such a sequence exists. An obvious orthogonalization process whose mildly complicated description will be omitted makes it possible to replace the sequence by another having properties (a), (b), (c) and (d) and in addition the following: For all y for which (c) is satisfied the $f_j(y)$ with j not greater than the dimension of $\mathcal{H}({}^1L^y)$ form a complete orthonormal set and the remaining $f_j(y)$ are zero. Choose a Borel set N in S such that $\mu(N)=0$, such that the $f_j(y)$ have $\mathcal{H}(L^y)$ as their closed linear span for all y in $S-N$ and such that $y \to (L^y_x(f_j(y)), f_k(y))$ is a Borel function on $S-N$ for all j, k, and x. (This is possible even though x varies over a noncountable set. We need only choose a countable dense subset of $\mathcal{W}$ and work with that instead. If the indicated functions are Borel functions for the members of this dense subset they will be so also for all x in $\mathcal{W}$.) For each $y \in S-N$ let $n(y)$ denote the dimension of $\mathcal{H}(L^y)$ and let ${}^yV$ denote the unitary map of $\mathcal{H}_n$ on $\mathcal{H}(L^y)$ which takes $c_1, c_2, \cdots$ into $c_1f_1(y) + c_2f_2(y) + \cdots$. Let ${}^1L^y_x = {}^yV^{-1}L^y_x{}^yV$ for all $x, y \in \mathcal{W} \times S-N$. Then $y \to {}^1L^y$ is a Borel function from $S-N$ to $\mathcal{W}^c$ and ${}^1L^y \simeq L^y$ for all $y \in S-N$. Thus $y \to (L^y)^\Delta$ is integrable as was to be proved.

It follows from the theorem just proved that the definition of direct integral presented here is equivalent to that discussed in §9 of [11] not only when $\mathcal{W}$ is a group but also when $\mathcal{W}$ is an algebra provided that the other definition is adapted in the obvious way to apply to algebras instead of groups. In particular the theory outlined in §9 of [11] and §2 of [12] applies in the present context and will be freely used in what follows. We note that the proof of Theorem 10.1 fills a gap in §9 of [11]. It was there carelessly asserted to be obvious.

We shall say that a representation L of $\mathcal{W}$ is *multiplicity free* if the ring $\mathcal{R}(L, L)$ of all intertwining operators of L with L is commutative and that it is of *type* I if it is a direct sum of multiplicity free representations. When $\mathcal{W}$ is a finite group or a finite dimensional algebra L is multiplicity free if and only if it is a direct sum of mutually inequivalent irreducible representations. Moreover in this case every representation of $\mathcal{W}$ is of type I. In the general case a representation is of type I if and only if the von Neumann-Murray operator ring generated by the L_x is of type I. (Cf. [12, §1]). For many, but by no means all, locally compact groups and Banach *-algebras it can be shown that every representation is of type I. We shall refer to such groups and algebras as being themselves of type I.

As indicated in §1 of [12] the theory of operator rings can be used to reduce the study of type I representations to the study of multiplicity free representations. We shall devote the rest of this section to a discussion of the extent to which the study of the multiplicity free representations of a given $\mathcal{W}$ may be reduced to the study of the measure classes in $\widehat{\mathcal{W}}$. If C is any standard measure class in $\widehat{\mathcal{W}}$ and μ is a finite member of C then by Theorem 10.2 the identity map $y \rightarrow L^y$ of $\widehat{\mathcal{W}}$ into $\widehat{\mathcal{W}}$ is integrable and we may form $\int_{\widehat{\mathcal{W}}} L^y d\mu(y)$ getting a well defined member M of $\mathcal{W}^r$. M depends only upon C and we shall denote it by $\mathcal{L}(C)$. Our principle concern is the extent to which $\mathcal{L}$ is a one-to-one map of the standard measure classes in $\widehat{\mathcal{W}}$ onto the multiplicity free members of $\mathcal{W}^r$. We shall see that the range of $\mathcal{L}$ is not always even included in the set of multiplicity free members of $\mathcal{W}^r$. One can also proceed in the reverse direction and assign a measure class $C = \mathcal{C}(L)$ to each multiplicity free member L of $\mathcal{W}^r$. As shown by the work of Mautner and its later refinements by others every representation L of $\mathcal{W}$ may be put in the form $\int_S L^y d\mu(y)$ where the L^y are irreducible and S is standard. This decomposition is badly nonunique in the general case but this nonuniqueness is essentially due to the possible existence of "inequivalent" maximal commutative subalgebras of $\mathcal{R}(L, L)$ and disappears for the most part when L is multiplicity free so that $\mathcal{R}(L, L)$ is commutative. The exact situation is described in Theorems 2.1 to 2.5 of [12]. In particular if ν is the Borel measure in $\widehat{\mathcal{W}}$ obtained by setting $\nu(E) = \mu(E')$ where E' is the inverse image of E in S under the mapping $y \rightarrow L^y$ then the measure class C is uniquely determined by the equivalence class of L. This is the measure class which we denote by $\mathcal{C}(L)$ (or $\mathcal{C}(L^\Delta)$). We turn now to a detailed study of the mappings $\mathcal{C}$ and $\mathcal{L}$ and the extent to which they invert one another.

We should like to show that $\mathcal{L}(\mathcal{C}(L)) = L$ and $\mathcal{C}(\mathcal{L}(C)) = C$ for all multiplicity free L and all standard measure classes C. However $\mathcal{L}(C)$ is not always multiplicity free and no proof exists that $\mathcal{C}(L)$ is always standard. Thus the left hand members of these equations need not even have a meaning. The next two theorems assert that when they do have a meaning the equations hold. Theorem 10.5 says a little more; that $\mathcal{L}(C)$ is multiplicity free whenever it is of type I.

THEOREM 10.4. *If L is multiplicity free and $\mathcal{C}(L)$ is standard then* $\mathcal{L}(\mathcal{C}(L)) = L^{\Delta}$.

Proof. Let $L^{\Delta} = \int_S L^y d\mu(y)$ where μ is a finite Borel measure in S, S is standard and the L^y are in $\widehat{\mathcal{W}}$. Let N be a Borel subset of $\widehat{\mathcal{W}}$ such that N is a null set with respect to $\mathcal{C}(L)$ and $\widehat{\mathcal{W}} - N$ is standard. If $y \to L^y$ is one-to-one it sets up a Borel isomorphism between the standard Borel spaces $\widehat{\mathcal{W}} - N$ and $S - N'$ where N' is the inverse image of N in S (Theorem 3.2). It follows at once that $\mathcal{L}(\mathcal{C}(L)) = L^{\Delta}$. If $y \to L^y$ is not one-to-one we consider the equivalence relation in $S - N'$ which it defines. Since $\widehat{\mathcal{W}} - N$ is countably separated the equivalence relation is measurable and we may apply Lemma 11.1 of [11] on quotient measures. This lemma tells us that if $\tilde{\mu}$ is the image of μ in $\widehat{\mathcal{W}} - \mathfrak{N}$ then for each member M of $\widehat{\mathcal{W}} - N$ there exists a finite Borel measure μ_M in the equivalence class in $S - N'$ defined by M such that $\mu = \int \mu_M d\tilde{\mu}(M)$ in a sense made precise in the statement of the lemma. It now follows from Theorem 2.11 of [12] that

$$L^{\Delta} = \int_{\widehat{\mathcal{W}} - N} \int_{S - N'} L^y d\mu_M(y) d\tilde{\mu}(M).$$

Let us denote $\int_{S-N'} L^y d\mu_M(y)$ by L^M. Now $L^y = M$ for μ_M almost all y. Hence, as is easy to see using Theorem 10.1, L^M is the direct sum of M with itself n_M times for some $n_M = 1, 2, \cdots$. Now by Theorem 2.8 of [12] n_M is measurable as a function of M. Hence either n_M is equal to one for almost all M or else there will exist a subrepresentation of L of the form $nK = K + K + \cdots + K$ to n terms where $n > 1$. This latter alternative is inconsistent with the commutativity of $\mathcal{R}(L, L)$. Thus $L^{\Delta} = \int_{\widehat{\mathcal{W}} - N} M d\tilde{\mu}(M) = \mathcal{L}(\mathcal{C}(L))$.

COROLLARY 1. *$\mathcal{C}$ restricted to those multiplicity free representations for which $\mathcal{C}(L)$ is standard is a one-to-one function*

COROLLARY 2. *If $\mathcal{W}$ has a metrically smooth dual then $\mathcal{C}$ is one to one and $\mathcal{L}$ is a left inverse for $\mathcal{C}$.*

THEOREM 10.5. *If C is standard and $\mathcal{L}(C)$ is of type* I *then $\mathcal{L}(C)$ is multiplicity free and $\mathcal{C}(\mathcal{L}(C)) = C$.*

Proof. Let μ be a finite member of C and let N be a μ null set of $\widehat{\mathcal{W}}$ such that $\mathcal{W} - N$ is standard. Let $y \to L^y$ be the identity mapping of $\widehat{\mathcal{W}} - N$ onto itself. Then $\mathcal{L}(C) = \int_{\widehat{\mathcal{W}} - N} L^y d\mu(y)$. Let P be the projection valued measure associated with this direct integral as described on p. 200 of [12]. Since the L^y are all irreducible the range $\mathcal{B}$ of this projection valued measure is a maximal σ Boolean algebra of projections in the corresponding Hilbert space. Let

$\mathcal{B}_1$ be the subalgebra of consisting of all $E \in \mathcal{B}$ such that the subrepresentations L^E and L^{1-E} of $L = \mathfrak{L}(C)$ obtained by restricting to the invariant ranges of E and $1-E$ are *disjoint* in the sense that $\mathcal{R}(L^E, L^{1-E}) = 0$. By Lemma 1.1 of [12] $\mathcal{B}_1$ is the intersection of $\mathcal{B}$ with the set of all projections in the center of $\mathcal{R}(L, L)$. Since $\mathcal{B}$ is maximal $\mathcal{B}_1$ is exactly the set of all projections in the center of $\mathcal{R}(L, L)$. Let $P_{E_1}, P_{E_2}, \cdots$ generate $\mathcal{B}_1$ where $E_1, E_2, \cdots$ are Borel subsets of $\widehat{\mathcal{W}} - N$. Let ϕ_{E_j} be the characteristic function of E_j and let Y be the image of $\widehat{\mathcal{W}} - N$ under the mapping $y \rightarrow \phi_{E_1}(y), \phi_{E_2}(y), \cdots$ in the Cartesian product of countably many replicas of the space consisting of 0 and 1. The mapping ϕ of $\widehat{\mathcal{W}} - N$ on Y defines a measurable equivalence relation in $\widehat{\mathcal{W}} - N$ to which we may apply Lemma 11.1 of [11] and then Theorem 2.11 of [12] just as in the proof of the preceding theorem. We arrive at a measure μ_z in each $\phi^{-1}(z)$ for each $z \in Y$ and a measure $\tilde{\mu}$ in Y such that

$$L = \int_Y \int_{\widehat{\mathcal{W}}-N} L^y d\mu_z(y) d\tilde{\mu}(z).$$

If Q is the projection valued measure associated with the outer integral then the range of Q is clearly $\mathcal{B}_1$. Hence by Theorems 2.5 and 2.6 of [12] the representations $\int_{\widehat{\mathcal{W}}-N} L^y d\mu_z(y)$ are almost all factor representations of type I. Hence by Theorem 2.7 of [12] the measure μ_z is concentrated in a point for almost all z. Thus $L = \int_Y L^z d\widehat{\mathcal{W}}(z)$ where the L^z are almost all irreducible. Hence $\mathcal{B}_1$ is maximal. Hence $\mathcal{B}_1 = \mathcal{B}$. Hence $\mathcal{R}(L, L)$ is commutative and L is multiplicity free. The fact that $\mathcal{C}(\mathfrak{L}(C) = C$ is now obvious.

COROLLARY 1. *$\mathfrak{L}$ restricted to those standard measure classes for which $\mathfrak{L}(C)$ is of type* I *is one-to-one.*

COROLLARY 2. *If $\mathcal{W}$ is of type* I *then $\mathfrak{L}$ is one-to-one and $\mathcal{C}$ is a left inverse for $\mathfrak{L}$.*

Combining the last two theorems we conclude the truth of

THEOREM 10.6. *The mapping $\mathfrak{L}$ sets up a one-to-one correspondence between the standard measure classes C in $\widehat{\mathcal{W}}$ such that $\mathfrak{L}(\mathcal{C})$ is of type* I *and the multiplicity free representations L in $\mathcal{W}^r$ such that $\mathcal{C}(L)$ is standard. The mapping $\mathcal{C}$ sets up the same correspondence in the reverse direction.*

COROLLARY. *If $\mathcal{W}$ is of type* I *and has a metrically smooth dual then the mutually inverse mappings $\mathfrak{L}$ and $\mathcal{C}$ set up a one-to-one correspondence between the multiplicity free representations in $\mathcal{W}^r$ and the measure classes in $\widehat{\mathcal{W}}$.*

Recall that two representations L and M are said to be disjoint if $\mathcal{R}(L, M) = 0$. In the reduction of the study of type I representations to the study of multiplicity free representations it is necessary to know when two multiplicity free representations are disjoint. The next theorem shows how disjointness of multiplicity free representations is related to a corresponding property for their associated measure classes. We say that two measure classes in the same Borel space S are disjoint if $S=S_1\cup S_2$ where S_1 and S_2 are Borel sets such that $S_1\cap S_2=0$ and such that each is a null set with respect to one of the measure classes.

THEOREM 10.7. *If C_1 and C_2 are disjoint standard measure classes such that $\mathfrak{L}(C_1)$ and $\mathfrak{L}(C_2)$ are of type* I *then $\mathfrak{L}(C_1)$ and $\mathfrak{L}(C_2)$ are disjoint representations. If L and M are multiplicity free representations which are disjoint and such that $\mathcal{C}(L)$ and $\mathcal{C}(M)$ are standard then $\mathcal{C}(L)$ and $\mathcal{C}(M)$ are disjoint measure classes.*

Proof. To prove the first statement let μ_1 and μ_2 be finite members of C_1 and C_2 respectively and let C_3 be the class of $\mu_1+\mu_2$. It is obvious that $\mathfrak{L}(C_3)$ is the direct sum of $\mathfrak{L}(C_2)$ and $\mathfrak{L}(C_1)$ and hence is of type I. By Theorem 10.5 then $\mathfrak{L}(C_3)$ is multiplicity free. Hence by Lemma 1.1 of [12] $\mathfrak{L}(C_1)$ and $\mathfrak{L}(C_2)$ are disjoint. To prove the second statement suppose that $\mathcal{C}(L)$ and $\mathcal{C}(M)$are not disjoint. Then there will exist a Borel subset F of $\widehat{\mathcal{W}}$ such that the restrictions of $\mathcal{C}(L)$ and $\mathcal{C}(M)$ to F are identical and not identically zero. Thus $\mathfrak{L}(\mathcal{C}(L))$ and $\mathfrak{L}(\mathcal{C}(M))$ have a subrepresentation in common and cannot be disjoint. Hence by Theorem 10.4 L and M cannot be disjoint. This completes the proof.

COROLLARY. *If $\mathcal{W}$ is of type* I *and has a metrically smooth dual then in the one to one correspondence set up by $\mathfrak{L}$ and $\mathcal{C}$ disjoint representations correspond to disjoint measure classes.*

For a $\mathcal{W}$ which is of type I and has a metrically smooth dual the corollaries to Theorems 10.6 and 10.7 combined with Theorem 1.4 of [12] reduce the problem of finding all members of $\mathcal{W}^r$ to the problem of finding all measure classes in $\widehat{\mathcal{W}}$ and in this sense solve the problem of describing all representations of $\mathcal{W}$ in terms of the irreducible ones. When $\widehat{\mathcal{W}}$ is countable or finite as happens when $\mathcal{W}$ is a compact group or a finite dimensional algebra the measure classes in $\widehat{\mathcal{W}}$ correspond one-to-one to the subsets of $\widehat{\mathcal{W}}$ and the solution alluded to is related in an obvious manner to the classical solution which describes a representation by giving the multiplicity with which each irreducible constituent occurs. In the general case, as we have pointed out in §6, a measure class in $\widehat{\mathcal{W}}$ may be regarded as a sort of generalized subset.

Because of the satisfactory way in which the "reduction to irreducibles" problem may be solved for $\mathcal{W}$'s which are of type I and have metrically smooth duals it is natural to ask just what groups and algebras have these properties. It follows from the theory of operator algebras that any $\mathcal{W}$ having only finite dimensional irreducible representations is of type I and we have already seen (corollary to Theorem 8.7) that every such $\mathcal{W}$ has a smooth dual. In particular every commutative $\mathcal{W}$ is of type I and has a smooth dual. As far as groups with infinite dimensional irreducible representations are concerned Harish-Chandra has shown that all connected semi-simple Lie groups are of type I. Moreover in view of Theorem 8.5 and the explicit results which are available on the classification of the irreducible representations of these groups it seems quite likely that all semi-simple Lie groups can be shown to have smooth duals as well. For solvable groups (Lie or otherwise) the situation is more complicated. In the article mentioned in the introduction, whose needs gave rise to this one, we study systematically the way in which the representations of group extensions are related to the representations of the normal subgroup and the quotient group. The results show that whether or not the extension is of type I or not depends rather delicately upon how the normal subgroup is imbedded and give methods of deciding in many special cases. Interestingly enough the conditions under which the extension may be shown to be of type I are the same as those under which we are able to given an explicit analysis of the irreducible representations of the extension. This fact suggests the conjecture, which is supported by other considerations that $\mathcal{W}$ is of type I if and only if its dual is metrically smooth. Whether or not this conjecture is true as stated it seems clear that the two conditions are very closely related. We hope to study the exact nature of this relationship more closely in a future article. In any event it is clear that there is a large and interesting class of groups and algebras for which both conditions hold.

When $\mathcal{W}$ fails to have a metrically smooth dual or to be of type I the question arises as to the extent to which $\mathcal{C}(L)$ can fail to be standard when L is multiplicity free and the extent to which $\mathcal{L}(C)$ can fail to be of type I when C is standard. As far as the first question is concerned it is conceivable that $\mathcal{C}(L)$ is always standard—we know of no counterexample. Our attempts to prove this however have not been successful. The problem may be regarded as that of showing that a representation which is multiplicity free in the sense of our global definition may be decomposed as a direct integral of *distinct* irreducible components. That this condition is necessary follows at once from Theorem 10.4. We prove now that it is sufficient.

THEOREM 10.8. *Let $y \rightarrow L^y$ be a one-to-one function from the standard Borel space S to $\widehat{\mathcal{W}}$ which is integrable with respect to a finite Borel measure μ in S. Then the measure ν in $\widehat{\mathcal{W}}$ into which μ is mapped by $y \rightarrow L^y$ is standard.*

Since $y \rightarrow L^y$ is integrable there exists a Borel function $y \rightarrow {}^1L^y$ from S to $\mathcal{W}^c$ such that ${}^1L^{y\Delta} = L^y$ for almost all y and by replacing S by a suitable Borel subset we may suppose that ${}^1L^{y\Delta} = L^y$ for all y. The mapping $y \rightarrow {}^1L^y$ is necessarily one-to-one and since $\mathcal{W}^c$ is standard it maps S isomorphically onto a Borel subset F of $\mathcal{W}^c$. Let $L^\Delta = \phi(L)$. Then the restriction ϕ_1 of ϕ to F is one-to-one. In order to prove the theorem it will clearly suffice to show that ϕ_1 is a Borel isomorphism of F with $\phi(F)$; that is that ϕ_1^{-1} is a Borel function. But for any Borel subset F' of F, $\phi_1(F)$ is a Borel set if and only if $\phi^{-1}(\phi_1(F'))$ is a Borel set. But $\phi^{-1}(\phi_1(F'))$ is the set of all representations in $\mathcal{W}^c$ of the form ULU^{-1} where U is unitary and $L \in F'$. Let us suppose for simplicity that all members of F act in the same $\mathcal{H}_n$. The argument for passing to the general case is simple and obvious. Let $\mathcal{G}_n$ be the standard Borel group of all unitary transformations of $\mathcal{H}_n$ onto $\mathcal{H}_n$ (cf. proof of Theorem 8.5). Then $\phi^{-1}(\phi_1(F'))$ is the image of $F' \times \mathcal{G}_n$ under the Borel map $L, U \rightarrow ULU^{-1}$. If this mapping were one-to-one $\phi^{-1}(\phi_1(F'))$ would be a Borel set by Theorem 3.2 since F', $\mathcal{G}_n$ and $\mathcal{W}^c$ are all standard. Now $ULU^{-1} = VMV^{-1}$ if and only if VU^{-1} sets up an equivalence between L and M. Since no two distinct members of F are equivalent $ULU^{-1} = VMV^{-1}$ if and only if $L = M$ and VU^{-1} is a multiple of the identity. Suppose that we can find a Borel subset $\mathcal{G}_n'$ of $\mathcal{G}_n$ such that $\mathcal{G}_n'$ contains one and only one member of each one parameter family $\exp(i\theta)V$ where θ varies from 0 to 2π. Then $\phi^{-1}(\phi_1(F'))$ will be the image of $F' \times \mathcal{G}_n'$ under a one-to-one Borel map, the argument just described will apply and the proof of the theorem will be complete. Such a Borel set is easily constructed as follows. Let $e_1, e_2, \cdots$ be the canonical basis in $\mathcal{H}_n$ and let V be in $\mathcal{G}_n'$ if and only if the first nonvanishing $(V(e_i), e_j)$ is real and positive where the pairs e_i, e_j have been ordered in some convenient fashion. The verification that $\mathcal{G}_n'$ has the required properties is obvious and will be left to the reader.

Concerning the second question we remark only that $\mathcal{L}(C)$ in fact need not be of type I even when C is standard. Indeed as shown by an example on pp. 590 and 591 of [10] a representation of type II can be a direct integral of distinct irreducible representations. On the other hand it follows from the theorem just proved that a direct integral of distinct irreducible representations is always of the form $\mathcal{L}(C)$ where C is a standard measure class. It would be interesting to have an intrinsic characterization of the standard measure classes for which $\mathcal{L}(C)$ is of type I.

We conclude with some brief remarks concerning work by other authors related to the results of this section. The prototype for these results is the now classical theory of Hahn and Hellinger describing unitary equivalence classes of self adjoint operators in terms of measure classes on the line. It has been recognized for some time by a number of mathematicians that the Hahn-Hellinger theory has an immediate and obvious extension to a theory classifying the equivalence classes of * representations of a commutative separable Banach * algebra $\mathcal{A}$ in terms of measure classes in its space $\widehat{\mathcal{A}}$ of regular maximal ideals. Recently H. Gonshor in his thesis [3] showed that the Hahn-Hellinger theory could be extended so as to apply to the "binormal" operators of A. Brown [1]. Our theory of course includes the commutative Banach algebra theory in the separable case and through it the Hahn-Hellinger theory. Applied to suitable noncommutative Banach algebras it can be made to include the results of Gonshor and to yield similar results for other classes of non-normal operators. The general problem of classifying representations of algebras (and hence of groups) has recently been attacked from a rather different point of view by Kadison and the results announced in a very interesting note [4]. His work is more general than ours in that he makes no separability assumptions and does not make any restrictions corresponding to our smoothness and type I conditions. On the other hand he does not, as is necessary for our purposes, describe his general representations in terms of irreducible ones. Instead he describes them in terms of the elementary positive definite functions on the algebra and certain limits of such functions. In order to obtain results of the sort that we need from Kadison's results it would be necessary to study the effect on his invariants of restricting them to the space of elementary positive definite functions and then passing to the quotient space obtained by identifying two elementary positive definite functions whenever they define the same irreducible representation. It is not clear how easy it would be to carry out such a study or to how much of an improvement of our results it might lead. It is possible and even likely that nonsmoothness in the indicated quotient space would play an interfering role just as it did in the formation of $\widehat{\mathfrak{A}}$ from the irreducible members of $\mathfrak{A}^c$. In any event it does not seem likely that one could thus obtain a simpler derivation of our results—even supposing given the results of [4].

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